1 Number, Algebra and Geometry

Chapter 1 Contents

1.1 Introduction

Mathematics plays an important role in our lives. It is used in everyday activities from buying food to organizing maintenance schedules for aircraft. Through applications developed in various cultural and historical contexts, mathematics has been one of the decisive factors in shaping the modern world. It continues to grow and to find new uses, particularly in engineering and technology, from electronic circuit design to machine learning.

Mathematics provides a powerful, concise and unambiguous way of organizing and communicating information. It is a means by which aspects of the physical universe can be explained and predicted. It is a problem-solving activity supported by a body of knowledge. Mathematics consists of facts, concepts, skills and thinking processes – aspects that are closely interrelated. It is a hierarchical subject in that new ideas and skills are developed from existing ones. This sometimes makes it a difficult subject for learners who, at every stage of their mathematical development, need to have ready recall of material learned earlier.

In the first two chapters we shall summarize the concepts and techniques that most students will already understand and we shall extend them into further developments in mathematics. There are four key areas of which students will already have considerable knowledge.

- numbers
- algebra
- geometry
- functions

These areas are vital to making progress in engineering mathematics (indeed, they will solve many important problems in engineering). Here we will aim to consolidate that knowledge, to make it more precise and to develop it. In this first chapter we will deal with the first three topics; functions are considered next (see Chapter 2).

1.2 Number and arithmetic

1.2.1 Number line

Mathematics has grown from primitive arithmetic and geometry into a vast body of knowledge. The most ancient mathematical skill is counting, using, in the first instance, the natural numbers and later the integers. The term **natural numbers** commonly refers to the set $\mathbb{N} = \{1, 2, 3, \dots\}$, and the term **integers** to the set $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$. The integers can be represented as equally spaced points on a line called the **number line** as shown in Figure 1.1. In a computer the integers can be stored exactly. The set of all points (not just those representing integers) on the number line represents the **real numbers** (so named to distinguish them from the complex numbers, which are

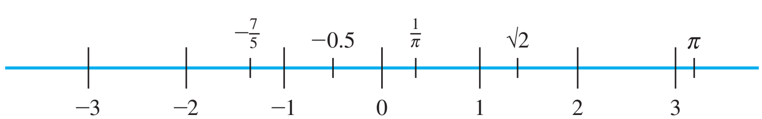


Figure 1.1
The number line.

discussed in Chapter 3). The set of real numbers is denoted by $\mathbb{R}$. The general real number is usually denoted by the letter x and we write 'x in $\mathbb{R}$', meaning x is a real number. A real number that can be written as the ratio of two integers, like $\frac{3}{2}$ or $-\frac{7}{5}$, is called a **rational number**. Other numbers, like $\sqrt{2}$ and π, that cannot be expressed in that way are called **irrational numbers**. In a computer the real numbers can be stored only to a limited number of figures. This is a basic difference between the ways in which computers treat integers and real numbers, and is the reason why the computer languages commonly used by engineers distinguish between integer values and variables on the one hand and real number values and variables on the other.

1.2.2 Representation of numbers

For everyday purposes we use a system of representation based on ten **numerals**: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. These ten symbols are sufficient to represent all numbers if a **position notation** is adopted. For whole numbers this means that, starting from the right-hand end of the number, the least significant end, the figures represent the number of units, tens, hundreds, thousands, and so on. Thus one thousand, three hundred and sixty-five is represented by 1365, and two hundred and nine is represented by 209. Notice the role of the 0 in the latter example, acting as a position keeper. The use of a decimal point makes it possible to represent fractions as well as whole numbers. This system uses ten symbols. The number system is said to be 'to base ten' and is called the **decimal** system. Other bases are possible: for example, the Babylonians used a number system to base sixty, a fact that still influences our measurement of time. In some societies a number system evolved with more than one base, a survival of which can be seen in imperial measures (inches, feet, yards, …). For some applications it is more convenient to use a base other than ten. Early electronic computers used **binary** numbers (to base two); modern computers use **hexadecimal** numbers (to base sixteen). For elementary (pen-and-paper) arithmetic a representation to base twelve would be more convenient than the usual decimal notation because twelve has more integer divisors (2, 3, 4, 6) than ten (2, 5).

In a decimal number the positions to the left of the decimal point represent units (10^0), tens (10^1), hundreds (10^2) and so on, while those to the right of the decimal point represent tenths (10^{-1}), hundredths (10^{-2}) and so on. Thus, for example,

$$\begin{array}{cccccc} 2 & 1 & 4 & \cdot & 3 & 6 \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\ 10^2 & 10^1 & 10^0 & & 10^{-1} & 10^{-2} \end{array}$$

so

$$\begin{aligned} 214.36 &= 2(10^2) + 1(10^1) + 4(10^0) + 3(\tfrac{1}{10}) + 6(\tfrac{1}{100}) \\ &= 200 + 10 + 4 + \tfrac{3}{10} + \tfrac{6}{100} \\ &= \tfrac{21436}{100} = \tfrac{5359}{25} \end{aligned}$$

In other number bases the pattern is the same: in base b the position values are b^0, b^1, b^2, … and b^{-1}, b^{-2}, … . Thus in binary (base two) the position values are units, twos, fours, eights, sixteens and so on, and halves, quarters, eighths and so on. In hexadecimal (base sixteen) the position values are units, sixteens, two hundred and fifty-sixes and so on, and sixteenths, two hundred and fifty-sixths and so on.

Example 1.1

Write (a) the binary number 1011101_2 as a decimal number and (b) the decimal number 115_{10} as a binary number.

Solution

(a) $1011101_2 = 1(2^6) + 0(2^5) + 1(2^4) + 1(2^3) + 1(2^2) + 0(2^1) + 1(2^0)$

$= 64_{10} + 0 + 16_{10} + 8_{10} + 4_{10} + 0 + 1_{10}$

$= 93_{10}$

(b) We achieve the conversion to binary by repeated division by 2. Thus

$115 \div 2 =$	57	remainder 1	(2^0)
$57 \div 2 =$	28	remainder 1	(2^1)
$28 \div 2 =$	14	remainder 0	(2^2)
$14 \div 2 =$	7	remainder 0	(2^3)
$7 \div 2 =$	3	remainder 1	(2^4)
$3 \div 2 =$	1	remainder 1	(2^5)
$1 \div 2 =$	0	remainder 1	(2^6)

so that

$115_{10} = 1110011_2$

Example 1.2

Represent the numbers (a) two hundred and one, (b) two hundred and seventy-five, (c) five and three-quarters and (d) one-third in

(i) decimal form using the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9;

(ii) binary form using the figures 0, 1;

(iii) duodecimal (base twelve) form using the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, Δ, ε.

Solution

(a) two hundred and one

(i) = 2 (hundreds) + 0 (tens) and 1 (units) = 201_{10}

(ii) = 1 (one hundred and twenty-eight) + 1 (sixty-four) + 1 (eight) + 1 (unit) = 11001001_2

(iii) =1 (gross) + 4 (dozens) + 9 (units) = 149_{12}

Here the subscripts 10, 2, 12 indicate the number base.

(b) two hundred and seventy-five

(i) = 2 (hundreds) + 7 (tens) + 5 (units) = 275_{10}

(ii) = 1 (two hundred and fifty-six) + 1 (sixteen) + 1 (two) + 1 (unit) = 100010011_2

(iii) = 1 (gross) + 10 (dozens) + eleven (units) = $1\Delta\varepsilon_{12}$
(Δ represents ten and ε represents eleven)

(c) five and three-quarters

(i) = 5 (units) + 7 (tenths) + 5 (hundredths) = 5.75_{10}

(ii) = 1 (four) + 1 (unit) + 1 (half) + 1 (quarter) = 101.11_2

(iii) = 5 (units) + 9 (twelfths) = 5.9_{12}

(d) one-third

(i) = 3 (tenths) + 3 (hundredths) + 3 (thousandths) + … = $0.333\ldots_{10}$

(ii) = 1 (quarter) + 1 (sixteenth) + 1 (sixty-fourth) + … = $0.010101\ldots_2$

(iii) = 4 (twelfths) = 0.4_{12}

1.2.3 Rules of arithmetic

The basic arithmetical operations of addition, subtraction, multiplication and division are performed subject to the **Fundamental Rules of Arithmetic**. For any three numbers a, b and c:

(a1) the commutative law of addition

$$a + b = b + a$$

(a2) the commutative law of multiplication

$$a \times b = b \times a$$

(b1) the associative law of addition

$$(a + b) + c = a + (b + c)$$

(b2) the associative law of multiplication

$$(a \times b) \times c = a \times (b \times c)$$

(c1) the distributive law of multiplication over addition and subtraction

$$(a + b) \times c = (a \times c) + (b \times c)$$

$$(a - b) \times c = (a \times c) - (b \times c)$$

(c2) the distributive law of division over addition and subtraction

$$(a + b) \div c = (a \div c) + (b \div c)$$

$$(a - b) \div c = (a \div c) - (b \div c)$$

Here the brackets indicate which operation is performed first. These operations are called **binary** operations because they associate with every two members of the set of real numbers a unique third member; for example,

$$2 + 5 = 7 \quad \text{and} \quad 3 \times 6 = 18$$

Example 1.3 Find the value of $(100 + 20 + 3) \times 456$.

Solution Using the distributive law we have

$$\begin{aligned}(100 + 20 + 3) \times 456 &= 100 \times 456 + 20 \times 456 + 3 \times 456\\ &= 45\,600 + 9120 + 1368 = 56\,088\end{aligned}$$

Here 100×456 has been evaluated as

$$100 \times 400 + 100 \times 50 + 100 \times 6$$

and similarly 20×456 and 3×456.

This, of course, is normally set out in the traditional school arithmetic way:

$$\begin{array}{r} 456\\ \underline{123\ \times}\\ 1\,368\\ 9\,120\\ \underline{45\,600}\\ \underline{56\,088}\end{array}$$

Example 1.4 Rewrite $(a + b) \times (c + d)$ as the sum of products.

Solution Using the distributive law we have

$$\begin{aligned}(a + b) \times (c + d) &= a \times (c + d) + b \times (c + d)\\ &= (c + d) \times a + (c + d) \times b\\ &= c \times a + d \times a + c \times b + d \times b\\ &= a \times c + a \times d + b \times c + b \times d\end{aligned}$$

applying the commutative laws several times.

A further operation used with real numbers is that of **powering**. For example, $a \times a$ is written as a^2, and $a \times a \times a$ is written as a^3. In general the product of n a's where n is a positive integer is written as a^n. (Here the n is called the **index** or **exponent**.) Operations with powering also obey simple rules:

$$a^n \times a^m = a^{n+m} \qquad \textbf{(1.1a)}$$

$$a^n \div a^m = a^{n-m} \qquad \textbf{(1.1b)}$$

$$(a^n)^m = a^{nm} \qquad \textbf{(1.1c)}$$

From rule (1.1b) it follows, by setting $n = m$ and $a \neq 0$, that $a^0 = 1$. It is also convention to take $0^0 = 1$. The process of powering can be extended to include the fractional powers like $a^{1/2}$. Using rule (1.1c),

$$(a^{1/n})^n = a^{n/n} = a^1$$

and we see that

$$a^{1/n} = {}^n\sqrt{a}$$

the nth root of a. Also, we can define a^{-m} using rule (1.1b) with $n = 0$, giving

$$1 \div a^m = a^{-m}, \qquad a \neq 0$$

Thus a^{-m} is the reciprocal of a^m. In contrast with the binary operations $+$, $\times$, $-$ and $\div$, which operate on two numbers, the powering operation $(\ )^r$ operates on just one element and is consequently called a **unary** operation. Notice that the fractional power

$$a^{m/n} = ({}^n\sqrt{a})^m = {}^n\sqrt{(a^m)}$$

is the nth root of a^m. If n is an even integer, then $a^{m/n}$ is not defined when a is negative. When ${}^n\sqrt{a}$ is an irrational number then such a root is called a **surd**.

Numbers like $\sqrt{2}$ were described by the Greeks as **a-logos**, without a ratio number. An Arabic translator took the alternative meaning 'without a word' and used the Arabic word for 'deaf', which subsequently became **surdus**, Latin for deaf, when translated from Arabic to Latin in the mid-twelfth century.

Example 1.5

Find the values of

(a) $27^{1/3}$ (b) $(-8)^{2/3}$ (c) $16^{-3/2}$

(d) $(-2)^{-2}$ (e) $(-1/8)^{-2/3}$ (f) $(9)^{-1/2}$

Solution

(a) $27^{1/3} = {}^3\sqrt{27} = 3$

(b) $(-8)^{2/3} = ({}^3\sqrt{(-8)})^2 = (-2)^2 = 4$

(c) $16^{-3/2} = (16^{1/2})^{-3} = (4)^{-3} = \frac{1}{4^3} = \frac{1}{64}$

(d) $(-2)^{-2} = \dfrac{1}{(-2)^2} = \frac{1}{4}$

(e) $(-1/8)^{-2/3} = [{}^3\sqrt{(-1/8)}]^{-2} = [{}^3\sqrt{(-1)}/{}^3\sqrt{(8)}]^{-2} = [-1/2]^{-2} = 4$

(f) $(9)^{-1/2} = (3)^{-1} = \frac{1}{3}$

Example 1.6

Express (a) in terms of $\sqrt{2}$ and simplify (b) to (f).

(a) $\sqrt{18} + \sqrt{32} - \sqrt{50}$ (b) $6/\sqrt{2}$ (c) $(1 - \sqrt{3})(1 + \sqrt{3})$

(d) $\dfrac{2}{1 - \sqrt{3}}$ (e) $(1 + \sqrt{6})(1 - \sqrt{6})$ (f) $\dfrac{1 - \sqrt{2}}{1 + \sqrt{6}}$

Solution (a) $\sqrt{18} = \sqrt{(2 \times 9)} = \sqrt{2} \times \sqrt{9} = 3\sqrt{2}$

$\sqrt{32} = \sqrt{(2 \times 16)} = \sqrt{2} \times \sqrt{16} = 4\sqrt{2}$

$\sqrt{50} = \sqrt{(2 \times 25)} = \sqrt{2} \times \sqrt{25} = 5\sqrt{2}$

Thus $\sqrt{18} + \sqrt{32} - \sqrt{50} = 2\sqrt{2}$.

(b) $6/\sqrt{2} = 3 \times 2/\sqrt{2}$

Since $2 = \sqrt{2} \times \sqrt{2}$, we have $6/\sqrt{2} = 3\sqrt{2}$.

(c) $(1 - \sqrt{3})(1 + \sqrt{3}) = 1 + \sqrt{3} - \sqrt{3} - 3 = -2$

(d) Using the result of part (c), $\dfrac{2}{1 - \sqrt{3}}$ can be simplified by multiplying 'top and bottom' by $1 + \sqrt{3}$ (notice the sign change in front of the $\sqrt{\ }$). Thus

$$\frac{2}{1-\sqrt{3}} = \frac{2(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$$

$$= \frac{2(1+\sqrt{3})}{1-3}$$

$$= -1 - \sqrt{3}$$

(e) $(1 + \sqrt{6})(1 - \sqrt{6}) = 1 - \sqrt{6} + \sqrt{6} - 6 = -5$

(f) Using the same technique as in part (d) we have

$$\frac{1-\sqrt{2}}{1+\sqrt{6}} = \frac{(1-\sqrt{2})(1-\sqrt{6})}{(1+\sqrt{6})(1-\sqrt{6})}$$

$$= \frac{1-\sqrt{2}-\sqrt{6}+\sqrt{12}}{1-6}$$

$$= -(1 - \sqrt{2} - \sqrt{6} + 2\sqrt{3})/5$$

This process of expressing the irrational number so that all of the surds are in the numerator is called **rationalization**.

When evaluating arithmetical expressions the following rules of precedence are observed:

- the powering operation $(\ )^r$ is performed first
- then multiplication $\times$ and/or division $\div$
- then addition $+$ and/or subtraction $-$

When two operators of equal precedence are adjacent in an expression the left-hand operation is performed first. For example,

$$12 - 4 + 13 = 8 + 13 = 21$$

and

$$15 \div 3 \times 2 = 5 \times 2 = 10$$

The precedence rules are overridden by brackets; thus

$$12 - (4 + 13) = 12 - 17 = -5$$

and

$$15 \div (3 \times 2) = 15 \div 6 = 2.5$$

This order of precedence is commonly referred to as BODMAS/BIDMAS (meaning: brackets, order/index, multiplication, addition, subtraction).

Example 1.7 Evaluate $7 - 5 \times 3 \div 2^2$.

Solution Following the rules of precedence, we have

$$7 - 5 \times 3 \div 2^2 = 7 - 5 \times 3 \div 4 = 7 - 15 \div 4 = 7 - 3.75 = 3.25$$

1.2.4 Exercises

1. Find the decimal equivalent of 110110.101_2.

2. Find the binary and octal (base eight) equivalents of the decimal number 16 321. Obtain a simple rule that relates these two representations of the number, and hence write down the octal equivalent of 1011100101101_2.

3. Find the binary and octal equivalents of the decimal number 30.6. Does the rule obtained in Question 2 still apply?

4. Use binary arithmetic to evaluate

   (a) $100011.011_2 + 1011.001_2$

   (b) $111.10011_2 \times 10.111_2$

5. Simplify the following expressions, giving the answers with positive indices and without brackets:

   (a) $2^3 \times 2^{-4}$ (b) $2^3 \div 2^{-4}$ (c) $(2^3)^{-4}$

   (d) $3^{1/3} \times 3^{5/3}$ (e) $(36)^{-1/2}$ (f) $16^{3/4}$

6. The expression $7 - 2 \times 3^2 + 8$ may be evaluated using the usual implicit rules of precedence. It could be rewritten as $((7 - (2 \times (3^2))) + 8)$ using brackets to make the precedence explicit. Similarly rewrite the following expressions in fully bracketed form:

   (a) $21 + 4 \times 3 \div 2$

   (b) $17 - 6^{2+3}$

   (c) $4 \times 2^3 - 7 \div 6 \times 2$

   (d) $2 \times 3 - 6 \div 4 + 3^{2-5}$

7. Express the following in the form $x + y\sqrt{2}$ with x and y rational numbers:

   (a) $(7 + 5\sqrt{2})^3$ (b) $(2 + \sqrt{2})^4$

   (c) $\sqrt[3]{(7 + 5\sqrt{2})}$ (d) $\sqrt{(\tfrac{11}{2} - 3\sqrt{2})}$

8. Show that

   $$\frac{1}{a + b\sqrt{c}} = \frac{a - b\sqrt{c}}{a^2 - b^2 c}$$

   Hence express the following numbers in the form $x + y\sqrt{n}$ where x and y are rational numbers and n is an integer:

   (a) $\dfrac{1}{7 + 5\sqrt{2}}$ (b) $\dfrac{2 + 3\sqrt{2}}{9 - 7\sqrt{2}}$

   (c) $\dfrac{4 - 2\sqrt{3}}{7 - 3\sqrt{3}}$ (d) $\dfrac{2 + 4\sqrt{5}}{4 - \sqrt{5}}$

9. Find the difference between 2 and the squares of

   $$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}$$

   (a) Verify that successive terms of the sequence stand in relation to each other as m/n does to $(m + 2n)/(m + n)$.

   (b) Verify that if m/n is a good approximation to $\sqrt{2}$ then $(m + 2n)/(m + n)$ is a better one, and that the errors in the two cases are in opposite directions.

   (c) Find the next three terms of the above sequence.

1.2.5 Inequalities

The number line (Figure 1.1) makes explicit a further property of the real numbers – that of **ordering**. This enables us to make statements like 'seven is greater than two' and 'five is less than six'. We represent this using the comparison symbols

$>$, 'greater than'
$<$, 'less than'

It also makes obvious two other comparators:

$=$, 'equals'
$\neq$, 'does not equal'

These comparators obey simple rules when used in conjunction with the arithmetical operations. For any four numbers a, b, c and d:

$(a < b \text{ and } c < d)$	implies	$a + c < b + d$	**(1.2a)**
$(a < b \text{ and } c > d)$	implies	$a - c < b - d$	**(1.2b)**
$(a < b \text{ and } b < c)$	implies	$a < c$	**(1.2c)**
$a < b$	implies	$a + c < b + c$	**(1.2d)**
$(a < b \text{ and } c > 0)$	implies	$ac < bc$	**(1.2e)**
$(a < b \text{ and } c < 0)$	implies	$ac > bc$	**(1.2f)**
$(a < b \text{ and } ab > 0)$	implies	$\frac{1}{a} > \frac{1}{b}$	**(1.2g)**

Example 1.8

Show, without using a calculator, that $\sqrt{2} + \sqrt{3} > 2(\sqrt[4]{6})$.

Solution

By squaring we have that

$$(\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{2}\sqrt{3} + 3 = 5 + 2\sqrt{6}$$

Also

$$(2\sqrt{6})^2 = 24 < 25 = 5^2$$

implying that $5 > 2\sqrt{6}$. Thus

$$(\sqrt{2} + \sqrt{3})^2 > 2\sqrt{6} + 2\sqrt{6} = 4\sqrt{6}$$

and, since $\sqrt{2} + \sqrt{3}$ is a positive number, it follows that

$$\sqrt{2} + \sqrt{3} > \sqrt{(4\sqrt{6})} = 2(\sqrt[4]{6})$$

1.2.6 Modulus and intervals

The size of a real number x is called its modulus (or absolute value) and is denoted by $|x|$ (or sometimes by mod(x)). Thus

$$|x| = \begin{cases} x & (x \geqslant 0) \\ -x & (x < 0) \end{cases} \tag{1.3}$$

where the comparator $\geqslant$ indicates 'greater than or equal to'. (Likewise $\leqslant$ indicates 'less than or equal to'.)

Geometrically $|x|$ is the distance of the point representing x on the number line from the point representing zero. Similarly $|x - a|$ is the distance of the point representing x on the number line from that representing a.

The set of numbers between two distinct numbers, a and b say, defines an **open interval** on the real line. This is the set $\{x{:}a < x < b, x \text{ in } \mathbb{R}\}$ and is usually denoted by (a, b). (Set notation will be fully described later (see Chapter 6); here $\{x{:}P\}$ denotes the set of all x that have property P.) Here the double-sided inequality means that x is greater than a and less than b; that is, the inequalities $a < x$ and $x < b$ apply simultaneously. An interval that includes the end points is called a **closed interval**, denoted by $[a, b]$, with

$$[a, b] = \{x{:}a \leqslant x \leqslant b, x \text{ in } \mathbb{R}\}$$

Note that the distance between two numbers a and b might be either $a - b$ or $b - a$ depending on which was the larger. An immediate consequence of this is that

$$|a - b| = |b - a|$$

since a is the same distance from b as b is from a.

Example 1.9

Find the values of x so that

$$|x - 4.3| = 5.8$$

Solution

$|x - 4.3| = 5.8$ means that the distance between the real numbers x and 4.3 is 5.8 units, but does not tell us whether $x > 4.3$ or whether $x < 4.3$. The situation is illustrated in Figure 1.2, from which it is clear that the two possible values of x are -1.5 and 10.1.

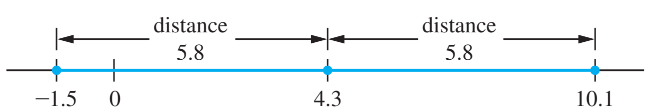


Figure 1.2 Illustration of $|x - 4.3| = 5.8$.

Example 1.10

Express the sets (a) $\{x{:}|x - 3| < 5, x \text{ in } \mathbb{R}\}$ and (b) $\{x{:}|x + 2| \leqslant 3, x \text{ in } \mathbb{R}\}$ as intervals.

Solution

(a) $|x - 3| < 5$ means that the distance of the point representing x on the number line from the point representing 3 is less than 5 units, as shown in Figure 1.3(a). This implies that

$$-5 < x - 3 < 5$$

Adding 3 to each member of this inequality, using rule (1.2d), gives

$$-2 < x < 8$$

and the set of numbers satisfying this inequality is the open interval $(-2, 8)$.

(b) Similarly $|x + 2| \leqslant 3$, which may be rewritten as $|x - (-2)| \leqslant 3$, means that the distance of the point x on the number line from the point representing -2 is less than or equal to 3 units, as shown in Figure 1.3(b). This implies

$$-3 \leqslant x + 2 \leqslant 3$$

Subtracting 2 from each member of this inequality, using rule (1.2d), gives

$$-5 \leqslant x \leqslant 1$$

and the set of numbers satisfying this inequality is the closed interval $[-5, 1]$.

It is easy (and sensible) to check these answers using spot values. For example, putting $x = -4$ in (b) gives $|-4 + 2| < 3$ correctly. Sometimes the sets $|x + 2| \leqslant 3$ and $|x + 2| < 3$ are described verbally as 'lies in the interval x equals -2 ± 3'.

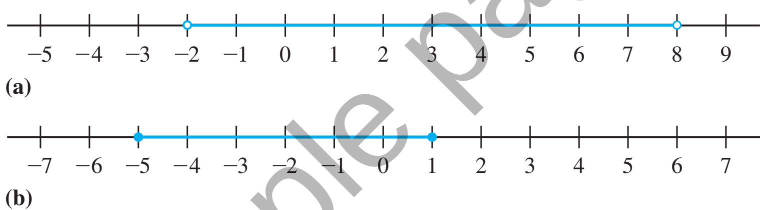


Figure 1.3
(a) The open interval $(-2, 8)$. (b) The closed interval $[-5, 1]$.

We note in passing the following results. For any two real numbers x and y:

$$|xy| = |x||y| \quad \textbf{(1.4a)}$$

$$|x| < a \text{ for } a > 0, \quad \text{implies} \quad -a < x < a \quad \textbf{(1.4b)}$$

$$|x + y| \leqslant |x| + |y|, \quad \text{known as the 'triangle inequality'} \quad \textbf{(1.4c)}$$

$$\tfrac{1}{2}(x + y) \geqslant \sqrt{(xy)}, \quad \text{when } x \geqslant 0 \text{ and } y \geqslant 0 \quad \textbf{(1.4d)}$$

Result (1.4d) is proved in Example 1.11 below and may be stated in words as

the arithmetic mean $\frac{1}{2}(x + y)$ of two positive numbers x and y is greater than or equal to the geometric mean $\sqrt{(xy)}$. Equality holds only when $y = x$.

Results (1.4a) to (1.4c) should be verified by the reader, who may find it helpful to try some particular values first, for example setting $x = -2$ and $y = 3$ in (1.4c).

Example 1.11 Prove that for any two positive numbers x and y, the arithmetic–geometric inequality

$$\tfrac{1}{2}(x + y) \geqslant \sqrt{(xy)}$$

holds.

Deduce that $x + \frac{1}{x} \geqslant 2$ for any positive number x.

We have to prove that $\frac{1}{2}(x + y) - \sqrt{(xy)}$ is greater than or equal to zero. Let E denote the expression $(x + y) - 2\sqrt{(xy)}$. Then

$$E \times [(x + y) + 2\sqrt{(xy)}] = (x + y)^2 - 4(xy)$$

(see Example 1.13)

$$\begin{aligned} E &= x^2 + 2xy + y^2 - 4xy \\ &= x^2 - 2xy + y^2 \\ &= (x-y)^2 \end{aligned}$$

which is greater than zero unless $x = y$. Since $(x + y) + 2\sqrt{(xy)}$ is positive, this implies

$E \geqslant 0$ or $\frac{1}{2}(x + y) \geqslant \sqrt{(xy)}$. Setting $y = \frac{1}{x}$, we obtain

$$\frac{1}{2}\left(x + \frac{1}{x}\right) \geqslant \sqrt{\left(x.\frac{1}{x}\right)} = 1$$

or

$$\left(x + \frac{1}{x}\right) \geqslant 2$$

1.2.7 Exercises

10 Show that $(\sqrt{5} + \sqrt{13})^2 > 34$ and determine without using a calculator the larger of $\sqrt{5} + \sqrt{13}$ and $\sqrt{3} + \sqrt{19}$.

11 Show the following sets on number lines and express them as intervals:

(a) $\{x: |x - 4| \leqslant 6\}$ (b) $\{x: |x + 3| < 2\}$

(c) $\{x: |2x - 1| \leqslant 7\}$ (d) $\{x: |\frac{1}{4}x + 3| < 3\}$

12 Show the following intervals on number lines and express them as sets in the form $\{x: |ax + b| < c\}$ or $\{x: |ax + b| \leqslant c\}$:

(a) $(1, 7)$ (b) $[-4, -2]$

(c) $(17, 26)$ (d) $[-\frac{1}{2}, \frac{3}{4}]$

13 Given that $a < b$ and $c < d$, which of the following statements are always true?

(a) $a - c < b - d$ (b) $a - d < b - c$

(c) $ac < bd$ (d) $\frac{1}{b} < \frac{1}{a}$

In each case either prove that the statement is true or give a numerical example to show it can be false.

If, additionally, a, b, c and d are all greater than zero, how does that modify your answer?

14 The average speed for a journey is the distance covered divided by the time taken.

(a) A journey is completed by travelling for the first half of the *time* at speed v_1 and the second half at speed v_2. Find the average speed v_a for the journey in terms of v_1 and v_2.

(b) A journey is completed by travelling at speed v_1 for half the *distance* and at speed v_2 for the second half. Find the average speed v_b for the journey in terms of v_1 and v_2.

Deduce that a journey completed by travelling at two different speeds for equal distances will take longer than the same journey completed at the same two speeds for equal times.

1.3 Algebra

The origins of algebra are to be found in Arabic mathematics as the name suggests, coming from the word *aljabara* meaning 'combination' or 're-uniting'. Algorithms are rules for solving problems in mathematics by standard step-by-step methods. Such methods were first described by the ninth-century mathematician Abu Ja'far Mohammed ben Musa from Khwarizm, modern Khiva on the southern border of Uzbekistan. The Arabic al-Khwarizm ('from Khwarizm') was Latinized to algorithm in the late Middle Ages. Often the letter x is used to denote an unassigned (or free) variable. It is thought that this is a corruption of the script letter ꝛ abbreviating the Latin word *res*, thing. The use of unassigned variables enables us to form mathematical models of practical situations as illustrated in the following example. First we deal with a specific case and then with the general case using unassigned variables.

The idea, first introduced in the seventeenth century, of using letters to represent unspecified quantities led to the development of algebraic manipulation based on the elementary laws of arithmetic. This development greatly enhanced the problem-solving power of mathematics – so much so that it is difficult now to imagine doing mathematics without this resource.

Example 1.12

A pipe has the form of a hollow cylinder as shown in Figure 1.4. Find its mass when

(a) its length is 1.5 m, its external diameter is 205 mm, its internal diameter is 160 mm and its density is 5500 kg m^{-3};

(b) its length is l m, its external diameter is D mm, its internal diameter is d mm and its density is ρ kg m^{-3}. Notice here that the unassigned variables l, D, d, ρ are pure numbers and do not include units of measurement.

Solution (a) Standardizing the units of length, the internal and external diameters are 0.16 m and 0.205 m respectively. The area of cross-section of the pipe is

$$0.25\pi(0.205^2 - 0.160^2)\ \text{m}^2$$

(*Reminder*: The area of a circle of diameter D is $\pi D^2/4$.)

Hence the volume of the material of the pipe is

$$0.25\pi(0.205^2 - 0.160^2) \times 1.5\ \text{m}^3$$

and the mass (volume × density) of the pipe is

$$0.25 \times 5500 \times \pi(0.205^2 - 0.160^2) \times 1.5\ \text{kg}$$

Evaluating this last expression by calculator gives the mass of the pipe as 106 kg to the nearest kilogram.

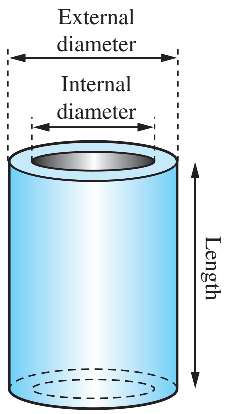


Figure 1.4 Cylindrical pipe of Example 1.12.

(b) The internal and external diameters of the pipe are $d/1000$ and $D/1000$ metres, respectively, so that the area of cross-section is

$$0.25\pi(D^2 - d^2)/1\,000\,000\ \text{m}^2$$

The volume of the pipe is

$$0.25\pi l(D^2 - d^2)/10^6\ \text{m}^3$$

Hence the mass M kg of the pipe of density ρ is given by the formulae

$$M = 0.25\pi\rho l(D^2 - d^2)/10^6 = 2.5\pi\rho l(D + d)(D - d) \times 10^{-5}$$

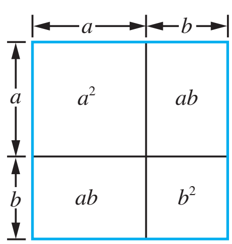
1.3.1 Algebraic manipulation

Algebraic manipulation made possible concise statements of well-known results, such as

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad \textbf{(1.5)}$$

Previously these results had been obtained by a combination of verbal reasoning and elementary geometry as illustrated in Figure 1.5.

Figure 1.5 Illustration of $(a + b)^2 = a^2 + 2ab + b^2$.



Example 1.13 Prove that

$$ab = \frac{1}{4}[(a + b)^2 - (a - b)^2]$$

Given $70^2 = 4900$ and $36^2 = 1296$, calculate 53×17.

Solution Since

$$(a + b)^2 = a^2 + 2ab + b^2$$

we deduce

$$(a - b)^2 = a^2 - 2ab + b^2$$

and

$$(a + b)^2 - (a - b)^2 = 4ab$$

and

$$ab = \frac{1}{4}[(a + b)^2 - (a - b)^2]$$

The result is illustrated geometrically in Figure 1.6. Setting $a = 53$ and $b = 17$, we have

$$53 \times 17 = \frac{1}{4}[70^2 - 36^2] = 901$$

This method of calculating products was used by the Babylonians and is sometimes called 'quarter-square' multiplication. It has been used in some analogue devices and simulators.

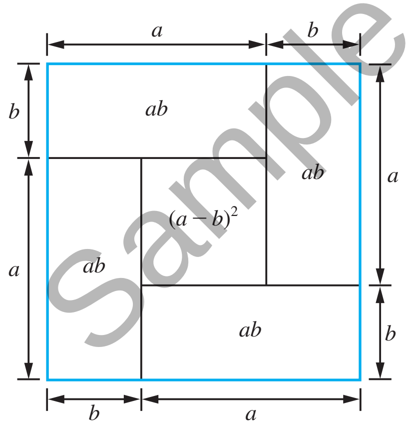


Figure 1.6 Illustration of $ab = \frac{1}{4}[(a + b)^2 - (a - b)^2]$.

Example 1.14 Show that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Solution Rewriting $a + b + c$ as $(a + b) + c$ we have

$$\begin{aligned}((a + b) + c)^2 &= (a + b)^2 + 2(a + b)c + c^2 \quad \text{using (1.5a)}\\ &= a^2 + 2ab + b^2 + 2ac + 2bc + c^2\\ &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac\end{aligned}$$

Example 1.15

Verify that

$$(x + p)^2 + q - p^2 = x^2 + 2px + q$$

and deduce that

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Solution

$$(x + p)^2 = x^2 + 2px + p^2$$

so that

$$(x + p)^2 + q - p^2 = x^2 + 2px + q$$

Working in the reverse direction is more difficult

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Comparing $x^2 + \frac{b}{a}x + \frac{c}{a}$ with $x^2 + 2px + q$, we can identify

$$\frac{b}{a} = 2p \quad \text{and} \quad \frac{c}{a} = q$$

Thus we can write

$$ax^2 + bx + c = a[(x + p)^2 + q - p^2]$$

where $p = \dfrac{b}{2a}$ and $q = \dfrac{c}{a}$

giving

$$\begin{aligned} ax^2 + bx + c &= a\left(x + \frac{b}{2a}\right)^2 + a\left(\frac{c}{a} - \frac{b^2}{4a^2}\right) \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

This algebraic process is called 'completing the square'.

We may summarize the results so far

$$(a + b)^2 = a^2 + 2ab + b^2 \tag{1.5a}$$

$$(a - b)^2 = a^2 - 2ab + b^2 \tag{1.5b}$$

$$a^2 - b^2 = (a + b)(a - b) \tag{1.5c}$$

$$a^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \tag{1.5d}$$

As shown in the previous examples, the ordinary rules of arithmetic carry over to the generalized arithmetic of algebra. This is illustrated again in the following example.

Example 1.16

Express as a single fraction

(a) $\frac{1}{12} - \frac{2}{3} + \frac{3}{4}$

(b) $\frac{1}{(x+1)(x+2)} - \frac{2}{x+1} + \frac{3}{x+2}$

Solution

(a) The lowest common denominator of these fractions is 12, so we may write

$$\frac{1}{12} - \frac{2}{3} + \frac{3}{4} = \frac{1-8+9}{12}$$
$$= \frac{2}{12} = \frac{1}{6}$$

(b) The lowest common multiple of the denominators of these fractions is $(x + 1)(x + 2)$, so we may write

$$\frac{1}{(x+1)(x+2)} - \frac{2}{x+1} + \frac{3}{x+2}$$
$$= \frac{1}{(x+1)(x+2)} - \frac{2(x+2)}{(x+1)(x+2)} + \frac{3(x+1)}{(x+1)(x+2)}$$
$$= \frac{1-2(x+2)+3(x+1)}{(x+1)(x+2)}$$
$$= \frac{1-2x-4+3x+3}{(x+1)(x+2)}$$
$$= \frac{x}{(x+1)(x+2)}$$

Example 1.17

Use the method of completing the square to manipulate the following quadratic expressions into the form of a number + (or −) the square of a term involving x.

(a) $x^2 + 3x - 7$ (b) $5 - 4x - x^2$

(c) $3x^2 - 5x + 4$ (d) $1 + 2x - 2x^2$

Solution

Remember $(a + b)^2 = a^2 + 2ab + b^2$.

(a) To convert $x^2 + 3x$ into a perfect square we need to add $(\frac{3}{2})^2$. Thus we have

$$x^2 + 3x - 7 = [(x + \tfrac{3}{2})^2 - (\tfrac{3}{2})^2] - 7$$
$$= (x + \tfrac{3}{2})^2 - \tfrac{37}{4}$$

(b) $5 - 4x - x^2 = 5 - (4x + x^2)$

To convert $x^2 + 4x$ into a perfect square we need to add 2^2. Thus we have

$$x^2 + 4x = (x + 2)^2 - 2^2$$

and

$$5 - 4x - x^2 = 5 - [(x + 2)^2 - 2^2] = 9 - (x + 2)^2$$

(c) First we 'take outside' the coefficient of x^2:

$$3x^2 - 5x + 4 = 3(x^2 - \tfrac{5}{3}x + \tfrac{4}{3})$$

Then we rearrange

$$x^2 - \tfrac{5}{3}x = (x - \tfrac{5}{6})^2 - \tfrac{25}{36}$$

so that $3x^2 - 5x + 4 = 3[(x - \tfrac{5}{6})^2 - \tfrac{25}{36} + \tfrac{4}{3}] = 3[(x - \tfrac{5}{6})^2 + \tfrac{23}{36}]$.

(d) Similarly

$$1 + 2x - 2x^2 = 1 - 2(x^2 - x)$$

and

$$x^2 - x = (x - \tfrac{1}{2})^2 - \tfrac{1}{4}$$

so that

$$1 + 2x - 2x^2 = 1 - 2[(x - \tfrac{1}{2})^2 - \tfrac{1}{4}] = \tfrac{3}{2} - 2(x - \tfrac{1}{2})^2$$

The reader should confirm that these results agree with identity (1.5d).

The number 45 can be factorized as $3 \times 3 \times 5$. Any product of numbers from 3, 3 and 5 is also a factor of 45. Algebraic expressions can be factorized in a similar fashion. An algebraic expression with more than one term can be factorized if each term contains common factors (either numerical or algebraic). These factors are removed by division from each term and the non-common factors remaining are grouped into brackets.

Example 1.18 Factorize $xz + 2yz - 2y - x$.

Solution There is no common factor to all four terms so we take them in pairs:

$$\begin{aligned} xz + 2yz - 2y - x &= (x + 2y)z - (2y + x) \\ &= (x + 2y)z - (x + 2y) \\ &= (x + 2y)(z - 1) \end{aligned}$$

Alternatively, we could have written

$$\begin{aligned} xz + 2yz - 2y - x &= (xz - x) + (2yz - 2y) \\ &= x(z - 1) + 2y(z - 1) \\ &= (x + 2y)(z - 1) \end{aligned}$$

to obtain the same result.

In many problems we are able to facilitate the solution by factorizing a quadratic expression $ax^2 + bx + c$ 'by hand', using knowledge of the factors of the numerical coefficients a, b and c.

Example 1.19 Factorize the expressions

(a) $x^2 + 12x + 35$ (b) $2x^2 + 9x - 5$

Solution (a) Since

$$(x + \alpha)(x + \beta) = x^2 + (\alpha + \beta)x + \alpha\beta$$

we examine the factors of the constant term of the expression

$$35 = 5 \times 7 = 35 \times 1$$

and notice that $5 + 7 = 12$ while $35 + 1 = 36$. So we can choose $\alpha = 5$ and $\beta = 7$ and write

$$x^2 + 12x + 35 = (x + 5)(x + 7)$$

(b) Since

$$(mx + \alpha)(nx + \beta) = mnx^2 + (n\alpha + m\beta)x + \alpha\beta$$

we examine the factors of the coefficient of x^2 and of the constant to give the coefficient of x. Here

$$2 = 2 \times 1 \text{ and } -5 = (-5) \times 1 = 5 \times (-1)$$

and we see that

$$2 \times 5 + 1 \times (-1) = 9$$

Thus we can write

$$(2x - 1)(x + 5) = 2x^2 + 9x - 5$$

It is sensible to do a 'spot-check' on the factorization by inserting a sample value of x, for example $x = 1$

$$(1)(6) = 2 + 9 - 5$$

Comment Some quadratic expressions, for example $x^2 + y^2$, do not have real factors.

The expansion of $(a + b)^2$ in (1.5a) is a special case of a general result for $(a + b)^n$ known as the binomial expansion. This is discussed again later (see Sections 1.3.6 and 7.7.2). Here we shall look at the cases for $n = 0, 1, \dots, 6$.

Writing these out, we have

$$(a + b)^0 = 1$$

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$