Essential Maths and Stats

for Higher Education

David Barton & David Cox

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Introduction

Essential Maths and Stats will be an invaluable resource for all students taking entry-level mathematics and statistics courses at New Zealand tertiary institutions. In addition, it is an ideal text to extend the abilities and understanding of secondary school students who are capable of extension work.

The scope of *Essential Maths and Stats* is from Year 10 secondary school mathematics right through to first and second year university mathematics and statistics papers. If a student is specialising in these subjects, he or she will find that this book provides a strong foundation from which to continue to degree level, either within New Zealand or internationally. If the student requires *some* mathematics or statistics in order to complete the requirements for a degree or diploma in another subject, or for a 'bridging' mathematics course, he or she will find that the essentials needed are all contained in this volume, and are explained clearly and without assumptions of prior knowledge of mathematics.

The topics at secondary level include, but are certainly not limited to, material for mathematics and statistics at NCEA (NZ) Levels 1 and 2 of the New Zealand mathematics curriculum (Years 11 and 12), as well as most of the material in both disciplines at Level 3 (Year 13).

In addition, the following topics at tertiary level are included:

- Algebra: matrices and vectors; the Taylor and Maclaurin series; hyperbolic functions
- Calculus: l'Hopital's rule; integration by parts; integration using partial fractions; integrating factors
- Statistics: *t*-distribution; hypothesis testing; chi-squared distribution; ANOVA.

The book is divided into six sections:

1. Numerical mathematics (clarifying the 'basics' of mathematics and outlining the essential tools)
2. Algebra 1
3. Trigonometry
4. Calculus
5. Algebra 2
6. Statistics (including probability).

Also included is a chapter on spreadsheets for those wishing to improve their computer skills.

Both secondary level and more advanced topics are presented in a clear and concise manner, to furnish the student with a solid mathematical foundation – he or she will then understand the methods involved and derive satisfaction from that understanding. Connections between different areas of mathematics are explained. The discussion is not overly theoretical, but rather emphasises the practical methods needed to achieve a mastery of each topic and a clear understanding of the process.

The chapters are designed so that they are essentially progressive: as far as possible one topic and one chapter lead onto the next. However, as the scope of mathematics and statistics is large, the student will very likely find him/herself jumping around the chapters, at least to a small extent, guided by the chapter headings.

David Barton
David Cox

1 The basics: integers, decimals, fractions, percentages

Galileo Galilei – If I were again beginning my studies, I would follow the advice of Plato and start with mathematics.

The essentials of mathematics are in numbers and in the rules by which they operate. Mathematics is, of course, logical and is also largely sequential – if you understand one concept or topic, the next will follow much more easily.

We begin with an explanation of these essentials of numeracy.

Integers

Here is the set of **integers**:

$I = \{\ldots, {}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, 4, 5, \ldots\}$

Integers include:

- the counting (or natural) numbers, $N = \{1, 2, 3, 4, \ldots\}$,
- zero $\{0\}$,
- the opposites of the counting numbers $\{{}^{-}1, {}^{-}2, {}^{-}3, {}^{-}4, \ldots\}$.

A useful way to think of integers is to use an **integer number line**. The further a number is to the right, the greater it is.

Adding and subtracting integers

We can show any integer addition or subtraction on a number line. Take the first integer as the starting point. The second integer tells us how far to move from the starting point to arrive at the answer.

- To add a positive integer, move right by that number of steps.
- To subtract a positive integer, move left by that number of steps.

Adding a negative integer is just the same as subtracting a positive one!

Example

Work out **a** ${}^{-}3 + 8$ **b** $5 - 12$ **c** $6 + {}^{-}14$

Answer

a ${}^{-}3 + 8 = 5$ **b** $5 - 12 = {}^{-}7$ **c** $6 + {}^{-}14 = {}^{-}8$

Also, subtracting a negative integer is just the same as adding a positive integer (the adjacent negative signs cancel each other out).

Example

$7 - {}^{-}10$

Answer

$7 - {}^{-}10 = 17$

Multiplying and dividing integers

Multiplication of integers is based on multiplication of whole numbers. If two integers with the *same* sign are multiplied, the result is *positive*.

Examples

$4 \times 8 = 32$ $\quad$ ${}^{-}5 \times {}^{-}2 = 10$

If two integers with *different* signs are multiplied, the result is *negative*.

Examples

${}^{-}10 \times 3 = {}^{-}30$ $\quad$ $6 \times {}^{-}1 = {}^{-}6$

The rules for division of integers are the same as for multiplication.

+ or ÷	Positive	Negative
Positive	Positive	Negative
Negative	Negative	Positive

Examples

$21 \div 3 = 7$ $\quad$ $42 \div {}^{-}6 = {}^{-}7$ $\quad$ ${}^{-}30 \div 5 = {}^{-}6$ $\quad$ ${}^{-}8 \div {}^{-}2 = 4$

Exercise 1.01

1 Evaluate these integer expressions.

- **a** $3 + {}^{-}2$
- **b** $3 - {}^{-}2$
- **c** ${}^{-}10 + 8$
- **d** ${}^{-}14 - {}^{-}3$
- **e** ${}^{-}24 \div {}^{-}6$
- **f** $1 \times {}^{-}4$
- **g** ${}^{-}2 - 8$
- **h** ${}^{-}4 - {}^{-}7$
- **i** ${}^{-}72 \div 6$
- **j** $35 \div {}^{-}7$
- **k** $60 \div {}^{-}2$
- **l** ${}^{-}12 + 1$
- **m** ${}^{-}5 - 2$
- **n** $6 \times {}^{-}4$

2 Calculate the answers to these integer expressions. $p = {}^{-}5$, $q = {}^{-}7$, $r = 35$, $s = {}^{-}3$.

- **a** $p + q$
- **b** $s - r$
- **c** rq
- **d** $r \div q$
- **e** $(p + q) \div s$
- **f** $p(q + s)$
- **g** $p - (q + s)$
- **h** $p - q + r - s$

3 A temperature of 2 °C drops by 6 °C. What is the new temperature?

4 Force can be calculated by multiplying the mass of an object (in kilograms) by its acceleration (in m/s^2), and is measured in Newtons (N). If an object is slowing down, its acceleration is negative. Calculate the braking force needed to slow down a car with mass 1175 kg at an acceleration of ${}^{-}3$ m/s^2. Comment on the meaning of the sign of your answer (+ or −).

Factors and multiples

The **factors** of a number are natural numbers that divide exactly into that number.

Example

The factors of 38 are 1, 2, 19 and 38.

The **multiples** of a number are the results when that number is multiplied by each of the natural numbers.

Example

The multiples of 7 are 7, 14, 21, 28, 35, ...

Prime numbers

A **prime number** has exactly *two* factors – the number itself, and 1. For example, 29 is a prime number – its only factors are 29 and 1. No other number divides exactly into 29.

1 is *not* a prime number, because it has only one factor.

Exercise 1.02

1 What is the largest number that is a factor of both these numbers?

- a 10 and 18
- b 24 and 9
- c 32 and 48
- d 60 and 75

2 Answer *True* or *False* for each of these statements about counting numbers.

- a Each number has at least two factors.
- b 1 is a factor of every number.
- c Every number is a factor of itself.
- d Each number has itself as a multiple.
- e There is no limit on how many multiples a number has.
- f If a number is smaller than another number, it must have fewer factors.

3 A retailer sells two kinds of vouchers: books ($25) and music ($40). Here are its sales figures for vouchers for one week recently. One of the figures is wrong.

Day	Mon	Tues	Wed	Thu	Fri
Total sales	$80	$70	$25	$200	$90

- a Which day must have incorrect sales figures?
- b Which day did the retailer sell only book vouchers? Explain.
- c Which day did the retailer sell only music vouchers?
- d On which day *must* the retailer have sold both items?

4 What is the largest two-digit prime number?

5 How many prime numbers are less than 30?

6 What is the smallest prime number that divides into 91?

7 How many *even* prime numbers are there?

8 What is the lowest number that is a multiple of both 6 and 8?

Decimals

Digits after a decimal point represent (in order) tenths, hundredths, thousandths, and so on.

Example

603.48 means 6 hundreds, 0 tens, 3 ones, 4 tenths and 8 hundredths.

To add or subtract decimals, make sure the decimal points are aligned. Write numbers with extra zeros if needed.

Example

1.3 – 0.52

$$\begin{array}{r} 1.30 \\ -\underline{0.52} \\ 0.68 \end{array}$$

When multiplying decimals, the number of digits after the decimal point in the answer is the same as the *total* number of digits after the decimal points in the original numbers.

Example

0.45 × 0.2

$$\begin{array}{rl} 0.45 & \text{(two digits after the decimal point)} \\ \times\ 0.2 & \text{(one digit after the decimal point)} \\ \hline 0.090 & \text{(three digits after the decimal point)} \end{array}$$

0.090 can be written as 0.09.

When dividing decimals without using a calculator, change the problem to one where you are dividing by a whole number. Do this by moving the decimal point the same number of places in both numbers.

Example

36.84 ÷ 0.5 has the same answer as 368.4 ÷ 5 (the decimal point moves 1 place)

$$\begin{array}{r} 7\,3.\,6\,8 \\ 5\overline{)\,3\,6\,{}^{1}8.\,{}^{3}4\,{}^{4}0} \end{array}$$

Exercise 1.03

1 Write these numbers as decimals.
 a 2 tens, 3 ones and 7 tenths
 b 9 tenths
 c 4 tenths and 1 hundredth
 d 5 hundredths

2 Write these numbers in order from smallest to largest.
 0.69, 0.609, 0.96, 0.06, 0.9, 0.0906

3 Do these addition/subtraction problems without using a calculator.
 a 7.2 + 6.9 b 0.05 + 0.6 c 7 − 0.34

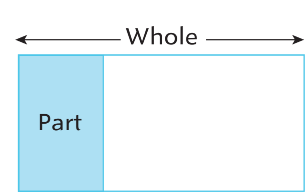
4 Do these multiplication and division problems without using a calculator.
 a 0.5 × 0.6 b 1.2 × 0.4 c 3.5 × 0.002
 d 7.12 ÷ 2 e 9.6 ÷ 0.4 f 0.56 ÷ 0.02

5 The voltage in a circuit changes from $^{-}2.7$ V to 4.6 V. How much has the voltage increased by?

6 The internal diameter of a bearing should be 23.1 mm, but when measured it is found to be 22.86 mm. What is the size of the error? Note: this is called the **absolute error**.

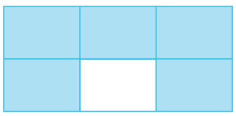
Fractions

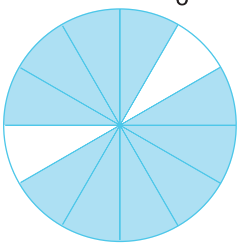
A fraction shows the relationship of part of an object to its whole. We write a fraction as $\frac{\text{part}}{\text{whole}}$.



We call the top number the **numerator**, and the bottom number the **denominator**.

Both of these diagrams show the fraction $\frac{5}{6}$.





This illustrates that in mathematics, we always express fractions in their simplest form, if possible.

Example

From the right-hand diagram above, $\frac{10}{12} = \frac{5 \times 2}{6 \times 2} = \frac{5}{6}$, because $\frac{2}{2}$ is another name for 1.

Recurring decimals

When some fractions are changed to decimals, we notice a repeating pattern rather than a decimal that terminates.

Example

$\frac{2}{5} = 0.4$ (terminates)

$\frac{1}{6} = 0.166\,666\,666\ldots$ (repeats)

We call decimals that repeat in a pattern **recurring decimals** and notate them by a dot over the first repeating digit. The table shows examples.

A single digit repeats	0.444 444 ...	Dot over that digit	$0.\dot{4}$
A pair of digits repeats	0.135 353 535 ...	Dots over that pair	$0.1\dot{3}\dot{5}$
A group of more than two repeats	0.908 908 908 ...	Dot over first and last digit in the group	$0.\dot{9}0\dot{8}$

When dividing by any number that has factors other than 2 and 5, a recurring decimal always results.

Example

Change $\frac{3}{11}$ to a decimal.

Answer

$\frac{3}{11} = 0.272\,727\,272\ldots$

$= 0.\dot{2}\dot{7}$

Exercise 1.04

1 Arrange these fractions in order from smallest to largest.

$\frac{2}{3}, \frac{3}{5}, \frac{13}{20}$

2 Multiply these fractions together.

a $\frac{2}{3} \times \frac{5}{8}$ b $\frac{1}{5} \times \frac{1}{4}$

3 Write the reciprocals of these fractions.

a $\frac{5}{8}$ b $\frac{1}{7}$

c $\frac{4}{19}$ d $\frac{-7}{5}$

4 Work out the reciprocals of these mixed numbers.

a $2\frac{1}{3}$ b $1\frac{7}{8}$

5 Divide these fractions.

a $\frac{2}{3} \div \frac{5}{6}$ b $\frac{4}{7} \div \frac{1}{2}$

6 Add or subtract these fractions.

a $\frac{3}{7} + \frac{1}{7}$ b $\frac{17}{30} - \frac{14}{30}$

7 These fractions have different denominators. Add or subtract them.

a $\frac{2}{3} + \frac{3}{4}$ b $\frac{7}{18} - \frac{5}{24}$

8 Multiply or divide these fractions.

a $\frac{4}{3} \times \frac{3}{5}$ b $\frac{4}{3} \div \frac{3}{5}$

9 Write these fractions as mixed numbers.

a $\frac{7}{2}$ b $\frac{41}{6}$

10 Change these mixed numbers to improper fractions.

a $1\frac{3}{8}$ b $6\frac{2}{3}$

11 Four pizzas are shared equally between the 10 players in a basketball team. What fraction of a pizza does each person get? Give your answer in its simplest form.

12 Two-fifths of the blood donors at a hospital have group O blood. Three-quarters of these are females. What fraction of all the donors are females with blood group O?

13 Here is a recipe for cooking rice for two people: add 3 cups of water to $\frac{3}{5}$ of a cup of rice and boil for 20 minutes.

a How many cups of water are needed to cook rice for five people?

b How many cups of rice are needed to cook rice for three people?

14 A formula used in physics is $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$.

a Calculate the value of f when $u = 4$ and $v = 7$.

b Express the formula *in words* without using the words/numerals 'divide' or 1.

15 An American drill size is $\frac{7}{64}$ inch. Convert this to millimetres, using the conversion factor of 25.4 mm in 1 inch. Give your answer correct to 2 dp.