CHAPTER 1

Approaches to mathematics teaching and learning

People are more likely to continue learning and using mathematics if they learn it with understanding and see its beauty and the possibility of applying it to matters that interest them, including games as well as more practical matters. Skill with the traditional basics is important to facilitate creative thinking about complex questions. However, skill alone is unlikely to prepare students for their future.

Those who have learned with understanding are more likely to remember the skills, to apply them efficiently and to be able to rediscover skills they may forget. They will be able to transfer their knowledge to new problems in the future and figure out mathematics for new situations. (Willoughby, 2010, p. 83)

CHAPTER OUTCOMES

This chapter will enable the reader to:

1.1 Describe how mathematical concepts and processes are best developed

1.2 Comprehend the significance of numeracy and its essential components

1.3 Explain the importance of using materials to develop mathematical concepts and processes

1.4 Recognise the role of language in teaching and learning mathematics

1.5 Realise the ways symbols represent mathematical ideas

1.6 Summarise the nature and purposes of assessment in mathematics

1.7 Recognise the value and role of diagnostic assessment

UNDERSTANDING THE WAYS in which mathematics is learned has changed. Once, the manner in which the content was organised was considered paramount and good teaching focused on ways of transmitting this preformed knowledge from teacher to learner. Content was categorised into a detailed syllabus, suggesting that an existing mathematics simply needed to be conveyed to children. Examples of each new idea were produced by the teacher, explained to the class, and followed up with practice in the form of worksheets or textbook pages. The role of the learner was to practise what was provided until it could be readily reproduced. Only then would the (successful) learner be shown and given practice in ways of applying this knowledge to different situations. In turn, the degree to which these procedures could be acquired and used determined the mathematical status of the individual learner and revealed the mathematical aptitudes with which they had been endowed.

When one aspect of the syllabus had been addressed to the satisfaction of the teacher, the next was introduced and the program proceeded throughout a year and across year levels. Yet, often children had not taken in the material as had been assumed. It is for this reason that assessment came to be seen as central to teaching. Only by determining how well an idea or procedure had been mastered could the next step in the learning sequence be introduced. Initially, the focus was on assessment at the completion of a teaching segment, but analysis of children working in classroom situations also came to be seen as crucial to building mathematical knowledge. This showed that the manner in which individual children interpreted what was presented to them was often quite different from that assumed when the content was organised. In this way, recognition of the significance of *diagnostic assessment*—where the underlying thinking being used became the focus of investigation, rather than the answers produced—gradually emerged as a significant part of the means to measure the success of a teaching approach or sequence. **Assessment** is now seen as a central component of teaching and learning—it needs to be an integral part of the planning used to build children's mathematical abilities.

LEARNING MATHEMATICS

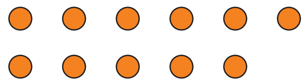
1.1

Describe how mathematical concepts and processes are best developed

An appropriately organised curriculum will always be important, but teaching is now more learner centred than content driven. In the first instance, this means that the organisation of ideas to be taught needs to derive as much from an understanding of how children learn as from the structure of the knowledge to be gained. New concepts and ways of thinking need to be linked to well understood existing knowledge, and their development needs to proceed in ways that reflect the manner in which the learner sees and makes sense of them, rather than in the order seen by someone who already has this understanding. The role of **materials** and models to build this thinking is paramount; they allow the learner to realise the significance of what is being introduced. (It is no coincidence that people who understand say, 'I *see* it now.') In turn, this use of materials and models can lead to a **language** that describes what is occurring before giving way to a more precise, mathematical terminology that encompasses and directs the meaning inherent in the mathematics. It is through talking about the ideas and seeing their representation with materials and models that learners are able to make the ideas their own. The scene is now set to introduce and establish the use of **symbols** as a concise way of representing the understanding of the concepts and processes that make up the mathematics being developed. It is this progression of ideas through the use of materials and models, to the language that describes and governs them, to the symbols that lead to powerful representations that can be manipulated in their own right that allows a mathematical way of thinking to be formed and applied to the problems of everyday life.

These considerations of the ways in which children learn led to a growing awareness that they do not simply take in mathematical knowledge that is merely transmitted to them, no matter how well organised and justified it is. Children are frequently observed to build their own ways of doing mathematics despite material or procedures introduced by a teacher. Sometimes this has led to alternative ways of coming to terms with mathematics and of using it to solve problems.

When asked how many counters there are,



a young child might answer 'Ten and 1 more is 11', or '4 and 4 is eight, 9, 10, 11', '2, 4, 6, 8, 10, 11', or other ways of grouping the counters rather than simply count '1, 2, 3, … 11'.

At other times, the way that children build their own ways of doing mathematics has produced consistent patterns of errors or misconceptions when place value is overlooked, renaming is neglected, and the significance of zero is not appreciated.

$$\begin{array}{r} 4\ 7 \\ +3\ 9 \\ \hline 716 \end{array}$$

While 7 and 9 has been correctly determined as *16* this has been written under the ones place, rather than renamed as *1 ten 6 ones*, which would give a correct answer of 86.

$$\begin{array}{r} 6\ 4 \\ -3\ 9 \\ \hline 3\ 5 \end{array}$$

In order for subtraction to take place, *64* needs to be renamed as 5 tens 14 ones to provide an answer of *25*. In this example, *4 ones* have been subtracted from *9 ones* to provide the incorrect answer, 35.

$$\begin{array}{r} 4\ 9 \\ \times 4\ 7 \\ \hline 16\ \ 63 \end{array}$$

While the multiplication of the ones digits and the tens digits has been correctly carried out, there is no understanding of renaming or of the need to 'cross places'. Instead, each 'column' has been treated as a multiplication on its own and the results written in the ones and tens places.

$$\begin{array}{r} 6\ \ \ 3 \\ 9\overline{)5\ 4\ 2\ 7} \end{array}$$

Rather than rename and use a remainder to determine the next step in the division process, the child has seen that *5 cannot be divided by 9* and moved to correctly divide *54 by 9 to get* 6. This has then been recorded above the 54, rather than in the hundreds place.

When the next digit, *2*, could not be divided by 9, instead of recording a 0 in the tens place to show this, the child has grouped the remaining digits together to get *27* and *divided this by 9 to give 3* which has been written above the 27, instead of in the ones place. This has resulted in an incorrect answer of *63 instead of 603*.

Learning that builds on the needs and knowledge of individual students also parallels the way in which mathematics evolved as people tried to come to terms with and make sense of problems in their everyday lives. In the early years of school, concepts in number, measurement, geometry, statistics and probability have always been developed in a similar manner through story situations from the children's own lives. Thus, addition and subtraction concepts and facts have grown from realistic embodiments, rather than through exercises in acquiring the addition and subtraction symbols + and – and their use with number symbols. This emphasis on problem situations out of which mathematics can grow is essential all the way through a student's schooling. As problems are understood and reconciled, the mathematics that is needed and that can develop from making sense of the solution process is personally developed and owned.

Learning mathematics, then, is necessarily an active process; the concepts and processes are too complex and the ideas often too abstract to allow them to be simply accepted through reading or telling. Children need to be involved in the formation of these new ways of thinking if they are to find them personally meaningful and be able to use them in different settings and formats. Experiences with problematic situations are fundamental to the way in which concepts and processes are built up or acquired, and resources for assisting learning need to incorporate play, games, everyday situations and objects from the child's world as well as specialised materials that might be seen to embody mathematical ideas.

Learning as a social activity

Learning is also a **social activity**, and both the mathematics and the manner in which it is learned are influenced by the way children interact with each other and with their teachers. Children construct meaning not only through the experiences they have with materials and problems, but also through examining and reflecting on their own reasoning and the reasoning of others. Talking about ideas that are being generated, sharing ways of tackling tasks and resolving difficulties, and describing outcomes that arise are integral to learning mathematics. At the same time, existing conceptions, whether gained from everyday experiences or previous learning, guide the understanding and interpretation of any new information or situation that is met. This often results in resistance to adopting new forms of knowledge or an unwillingness to give up or adapt previously successful thinking. Indeed, old ways of thinking are not usually given up without resistance, and their replacement by, or extension to, new conceptions is guided by those that already exist. Consequently, the intuitive beliefs and methods of children may appear very different from accepted mathematical practice and may also be very resistant to change. Teachers will need to ask questions that challenge ill-formed ideas and inappropriate generalisations, and pose new problems that will require the revision of old constructions or ways of thinking.

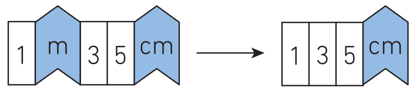
$$\begin{array}{r} 6.5 \\ +5.8 \\ \hline 12.3 \end{array}$$

Child: *With decimals, you have to line up the decimal points.*

$$\begin{array}{r} 6.5 \\ \times 5.8 \\ \hline 51.2 \\ 325.0 \\ \hline 376.2 \end{array}$$

The role of a teacher is often more as a facilitator of what is needed in order to become proficient; however, there will always be a place for showing how things are to be done. The challenge will then be to lead children to come to understand and accept this as a method of their own, rather than simply practising and acquiring by rote another person's way of doing something. Evidence from children who have experienced difficulty in learning mathematics has shown that those who simply acquired teacher taught techniques by rote were often unable to apply this knowledge and frequently forgot, or at least were unable to recall when needed, knowledge that had been earlier assumed to have been learned (Booker, 2011). In contrast, those who participated actively in their own learning were more able to use this knowledge and tended to maintain it for future use and adaptation.

1 m 35 cm can be renamed as 135 cm



Cooperative learning

Rather than working in isolation from other learners, it is often better for children to work cooperatively so as to encourage mathematical discussion and resolution of the various interpretations that emerge. Group activities and projects need to be organised to allow children to work on shared tasks, rather than have them perform individually on problems, worksheets, even pages from a text. In this way, they can work together in pairs or groups of 3 or 4, taking turns to record any working or observations in order to discuss them with the larger group later. Of course, at first one child may do more of the thinking and activity than the others, but whole class discussion about the mathematics of the situations on which they are working can then focus on the need of all the children to be able to talk about the activity. Indeed, judicious questioning of the child who watched more than participated can draw attention to the need to attend to all aspects of the task at hand and to voice uncertainties as they occur, rather than leave it to the more capable or dominant child. This allows a variety of ways of thinking about a particular situation to arise and add to the richness of the learning for all participants.

Cooperative learning can then go beyond merely working together on set tasks, and an atmosphere can be created in which children construct their own mathematical conceptions. The goals of learning, the discussions about the means of achieving those goals and the individual paths taken need to be at the centre of classroom learning. An attitude that each individual will reach their learning goal only if the others in the group also reach their goals is as important as the goals themselves. This is in distinct contrast to a classroom where learning is individualistic or competitive, with children working by themselves at their own pace to achieve goals unrelated to those of their classmates.

Learning cooperatively is also crucial in promoting children's ability to communicate and reason mathematically. The interactions between teachers and children, and especially among children, influence what is learned and how it is learned. In particular, attempts to communicate their thinking help to develop children's understanding that mathematics is conjectural in nature—that mathematical activity is concerned with reasoning about possibilities, rather than learning the results presented by others. Indeed, trying to make sense of methods and explanations they see or hear from others is fundamental to constructing mathematical meanings (Yackel et al., 1990). Within a framework of learning cooperatively, each child can be viewed both as an active reorganiser of their personal mathematical experiences and as a member of a community or group in which they actively participate in the continual regeneration of 'taken-as-shared' ways of doing mathematics (Cobb & Bauersfeld, 1995). Institutionalised practices such as using tens and ones in a place value sense, or following a particular method for measuring the area of a circle, can then emerge anew for each child, yet conform to accepted norms of mathematical behaviour.

Equity and diversity

Other factors in teaching and learning mathematics will need to be borne in mind while organising an overall program (see Chapter 11) and considering how to plan everyday activities. For instance, issues of *equity* and *diversity* impinge on efforts to 'make mathematics education accessible to all students' (Atweh, Vale & Walshaw, 2012). **Gender** issues have been of particular concern at all levels of education over the past three decades and much effort has been applied to achieve gender equity in education, especially in the fields of mathematics and science. The extent to which mathematics is perceived to be a 'male domain' seems to be waning and, particularly among younger students, girls' mathematical potential is no longer regarded as inferior to boys' (Forgasz & Leder, 2001). Consequently, there is now a growing acceptance that there is no longer a 'problem' for girls and mathematics, and the focus has switched to a perceived problem for boys. However, more males than females continue to study the most demanding mathematics courses offered, and males still dominate in mathematically related careers. Burton (2001) calls this 'the fable of the underachieving boy' and draws attention to the continuing need to encourage girls to participate in mathematics at all levels, and to engage with the science and technology courses that lead to more high status and influential STEM related careers.

Vale and Bartholomew (2008) reported that PISA surveys showed that boys are more often among the highest achievers, and that:

> there remains a difference in the ways that male and female students respond to their own mathematical experiences … boys reported higher levels of enjoyment, interest and self-efficacy in mathematics than girls, and boys more highly valued the use of technology in mathematics. (Vale & Bartholomew, 2008)

They also suggest that this leads to boys being more likely to enrol in higher level mathematics than girls, because of the widely reported positive relationship between affective factors and enrolment. In particular, this has led Forgasz (2010) and Vale (2010) to identify a widening gender gap favouring males in achievement in primary and secondary mathematics and participation at the senior secondary level, and to suggest that teachers need to continue to be aware of 'gender as a factor related to students' perceptions, participation and achievement in mathematics'.

Learning using instructional games

When learners participate in the playing of instructional games, the manipulation of materials and the verbalisation of actions, thoughts and interpretations assist in the construction of mathematical concepts. An element of chance ensures that each player has an opportunity to win and build self esteem. Games themselves are seen as fun, not only providing motivation but also ensuring the full engagement on which constructive learning depends (Blum & Yocom, 1996). Often this means that while children may not engage in learning to please a parent or teacher, and can rarely accept that mathematics will be useful in later life, they will willingly learn in order to participate with their peers in socially rewarding activities. In this way, instructional games can also contribute to teaching and learning by providing a background in which mathematical concepts can be developed and constructed along with a social situation that encourages cooperation and shared learning (Booker, 2000). At the same time, problem solving ability is improved when the discovery and use of strategies is required and previously acquired skills are maintained through motivating practice.

Further, the rule governed behaviour in these games is suggestive of the actions envisaged in the teaching and learning of mathematics. The match between the expectations for involvement in mathematical learning and the behaviours freely committed to game playing led to the inclusion of instructional games in mathematics programs. Initially, this use concentrated on practice aspects such as that required for basic fact learning, to provide motivation or to reward children for progress they have made. However, observations of games being played led to their extension to a wider range of concepts and processes (Larouche, Bergeron & Herscovics, 1984), and use can also be made of the interest generated by games to assist children to generalise to the more abstract recorded forms and higher level mathematical ideas. Indeed, Vygotsky (1978) has argued that 'the influence of play on a child's development is enormous', in that action and meaning can become separated and abstract thinking can thereby begin.

Above all, involvement in instructional games induces children to make sense of their ideas and the interpretations of others. The dialogue engaged in while playing facilitates the construction of mathematical knowledge, allowing the articulation and manipulation of each player's thinking. Such communication helps to extend a conceptual framework through a process of reflection and points to the central role of language, as the social interaction gives rise to genuine mathematical issues. In turn, these problems engender an exchange of ideas, with children striving to make sense of their mathematical activity, and lead them to see mathematics as a social process of sense making requiring the construction of consensual mathematical understandings.

Learning cooperatively and using instructional games can lead children to value persistence in working at a challenging task, in contrast to the mere repetition of similar exercises; to engage in meaningful activity in preference to procedures acquired by rote; and to see that cooperation and negotiation are productive at both a personal and social level. Consequently, the learning of mathematics can be viewed as an active, problem solving process in which social interactions help to promote understanding and reconcile the various interpretations and ways of thinking and acting that can arise.

Learning using apps, electronic resources and calculators

A quick search will uncover an abundance of apps, online activities, instructional videos and computer games, many of which provide engaging learning situations and enhance motivation (Wall, Beatty & Rogers, 2015; Harrison, 2018; Booker, 2002; 2004). However, it is important that these uses of technology go beyond simply providing review and practice of material presented in class to develop new ideas, strategies and approaches to problem solving. Choosing a quality app or program from among those offered commercially or through teaching websites can provide rich tasks for classroom use and lead to whole class discussion about the mathematics that is being developed. In this way, as Harrison (2018) points out, while '[a] good mathematical app does not replace a good mathematics teacher—it can, however, make good mathematics teaching easier'. But this can only occur if the mathematical ideas and processes presented within the app are in line with classroom and curriculum approaches and expectations.

The use of resources such as YouTube, tablets, spreadsheets, dynamic software and digital learning resources—for example, resolve (www.resolve.edu.au), topdrawer (topdrawer.aamt.edu.au), scootle (www.scootle.edu.au) and the national digital learning resources network (https://www.esa.edu.au)—continues to change the face of mathematics in today's classrooms (Olive & Makar, 2010; Borba et al., 2017). The range of materials available is expanding constantly, with access to lesson planning, development of particular mathematics ideas and processes, assessment items, and suggestions for teaching a range of mathematical topics. Videos of expert teachers engaging mathematics classes are another form of help readily available on many internet channels and mathematics teaching sites.

As the *Australian Curriculum: Mathematics* (www.australiancurriculum.edu.au) notes, teachers will need to incorporate explicit teaching of some of these resources, and this is particularly true of calculator use in the primary school, not only because of the many 'hidden' capabilities of these devices but also because of their use in standardised tests, including the National Assessment Program in Numeracy. The manner in which particular operations can be carried out most efficiently on a calculator is not obvious. Steps that with other methods need to be carried out one at a time and one after the other can be combined. Answers can be seen intuitively, rather than only at the end of a large number of separate steps. Methods can be quickly explored with the data of the problem, rather than analysed through logical or algebraic expressions, and answers can be readily put back into the problem to see if they make sense. Calculators also allow a teacher to introduce more realistic problems, sooner and more frequently than when only traditional computational skills are available. Since the world in which they will live is governed by the use of calculating and computing devices, children need to grow up with this technology forming as much a part of their classroom experiences as it is part of their world outside of school, seeing its use in developing ways of thinking and solving problems as a natural part of learning and using mathematics.

Calculators are, however, no more a simple replacement for other ways of doing things than they are obvious to use. Most adults who use calculators at work, home or for their leisure activities were self taught and consequently feel less than adequately competent with certain functions or less able to do certain computations efficiently. Effective ways of using this technology need to occur from the beginning years of school so that it is seen as simply one way among many to determine answers, another way of expressing results, and, above all, as a tool to which understanding and reasoning can be applied. In order to use a calculator in these ways, a number of mathematical understandings and skills are required. Recognition of the various symbols and the actions they represent (+, −, ×, ÷, =, √, %) is essential, as is knowing the concepts for these operations and the variety of meanings and situations in which they might be used. Familiarity with the operations in other contexts is necessary in order to know when to use them, to provide some awareness of likely outcomes, and to establish a measure of the reasonableness of results. Awareness of how the various memory functions operate, including the constant keys, is also needed to allow more efficient computation. A plan to 'see' what is happening when the calculator does not show it and for recording intermediate results is also beneficial. It is helpful to learn to use the nonwriting hand to key in numbers and operations and operate the calculator, so that the writing hand is available to write any outcome or important points along the way.

The activities developed throughout *Teaching Primary Mathematics* refer to suitable calculator activities, but the difficulty that always arises relates to the different key formats, operating procedures and capabilities. Familiarity with the calculators used in the classroom needs to be built up by the teacher and then explored jointly with the children. Calculators can be used as an integral part of the learning across numeracy topics, but it is important for children and their teachers to realise that a calculator needs to be seen as just one way of finding solutions among many. At times it will be easier to obtain an answer using mental calculations; at other times an approximate result would suffice; while an accurate solution may just as readily be found using pencil and paper as with a calculator. The most important end result is that each individual child should be in a position to choose the form of calculation best suited to a particular situation or task. At the same time, since calculators can do things at a greater speed, misunderstandings can come to light more quickly. It is much easier to make more mistakes, more frequently, on a calculator when you do not know what you are doing than it is with pencil and paper computations. The calculator can thus also be a tool to identify misconceptions and errors in a child's mathematical ways of thinking.

NUMERACY AND TEACHING MATHEMATICS

> To be truly numerate involves more than the acquisition of mathematical routines ... no matter how well they are learned.
>
> Students need to learn mathematics in ways that:
>
> - enable them to recognize when mathematics might help them to interpret information or solve practical problems
> - apply their knowledge appropriately in contexts where they will have to use their mathematical reasoning processes
> - choose mathematics that makes sense in the circumstances
> - make assumptions, resolve ambiguity and judge what is reasonable.
>
> (*National Numeracy Review Report*, DEEWR, 2008 p. xi)

1.2

Comprehend the significance of numeracy and its essential components

As technology becomes ever more central to all aspects of life, there are increasing calls for a more numerate population. It is no longer considered sufficient for children simply to study mathematics; they need to be able to *use* their mathematical knowledge in an ever broadening range of activities. Over 2 decades ago, Orrill (2001) remarked that society is 'awash in numbers' and 'drenched in data', and this has increased exponentially ever since thanks to the pervasive technology at the personal level as well as at work and study. For those comfortable with and competent in thinking about numbers, this provides a basis for evaluating such issues as the benefits and risks of medical treatments, estimates for budgets that will allow or disallow access to education and transport, and many other concerns that were once available only to specialists and those in the know. Conversely, individuals who lack an ability to think numerically will be disadvantaged and at the mercy of other people's interpretation and manipulation of numbers. Indeed, as Steen noted as long ago as 1997, 'an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg's time'. Expanding his analysis later, Steen highlighted that, in contrast to the study of further mathematics, numeracy is concerned with applying elementary ideas in sophisticated settings, rather than generalising these ideas to more abstract concepts and more complex processes.

> Mathematics thrives as a discipline and as a school subject because it was (and still is) the tool par excellence for comprehending ideas of the scientific age. Numeracy will thrive similarly because it is the natural tool for comprehending information in the computer age. (Steen, 2001, p. 111)

The term 'numeracy' (sometimes referred to as mathematical or quantitative literacy), concerned with using, communicating and making sense of mathematics in a range of everyday applications, emerged to provide a more satisfactory description of these extended mathematical processes and ways of thinking. An ability to explore, conjecture and reason logically, and to use a variety of mathematical methods to solve problems, is also fundamental, along with a capacity to understand information presented in mathematical terms so as to use and apply mathematical processes and communicate mathematically.

Positive attitudes towards *involvement* in mathematics, problem solving and applications, as well as a capacity to work systematically and logically, and to communicate with and about mathematics, are also central to being numerate. Mathematical communication abilities are essential in understanding and assessing the proposals of others, to convey arguments and justifications to a broader audience, and to analyse and interpret information. Discussing and writing about the mathematics that has been completed, and focusing on the thinking processes that were followed, the attempts that were made along the way, and the justification that allows particular approaches and solutions, can be used to build up an ability to read, write and speak with and about mathematics.

As the certainties of the past have given way to the uncertainties of the present and future, this also means that the formal techniques of number, measurement and geometry that gave exact and unalterable results must make room for ways to examine and explore less certain situations using statistics and probability, and include a range of estimation and approximation processes. Thus, numeracy should be seen to include the content of mathematics, particularly number sense, spatial sense, measurement sense, data sense and a feeling for chance, together with a focus on problem solving and the uses of mathematics in communication, as shown in the table below.

Numeracy overview

NUMERACY			
CONTENT	**+ PROBLEM SOLVING**	**+ SENSE MAKING**	**+ COMMUNICATION**
• number • measurement • geometry • statistics and probability • algebra	• analyse the problem • explore possible means to a solution	*number sense* • understanding of numeration and computation	• discuss and write about mathematics • reflection
	• select and try a solution process	*spatial sense* • visualisation of properties and relationships	• present arguments in mathematical form • interpret data presented graphically and statistically
Focus on • technology—calculators, apps, computers as tools to aid thinking • estimation vs exact • mental processes as well as recorded	• analyse solution and possible answer's sense in problem context	*measurement sense* • application of number and spatial sense *sense of statistics and probability* • inclination to use flexibility • personal strategies • ability to interpret	

Since 2008, the National Assessment Program in Numeracy has been used to determine the proportion of students who have achieved nationally agreed numeracy standards. From 2016, it has been aligned to the *Australian Curriculum: Mathematics* incorporating proficiency strands of conceptual understanding, fluency, problem solving and reasoning across the content strands of *Number and Algebra; Measurement and Geometry; Statistics and Probability.*

- *Understanding*—comprehension of mathematical concepts, operations and relations, and the connections among them—the 'why' as well as the 'how' of mathematics.

- *Fluency*—facility in carrying out processes flexibly, accurately, efficiently and appropriately; ready recall of factual knowledge.
- *Problem solving*—ability to formulate, represent and solve mathematical problems, and to communicate solutions effectively.
- *Reasoning*—capacity for logical thought, reflection, explanation, justification and proof—exploration, generalisation, description.

In particular, numeracy requires meaningful concepts for numeration and computation to provide a full understanding of whole numbers and the fraction ideas and negative numbers that extend them. In turn, these concepts allow fluent processes for place value, renaming, rounding, automatic access to basic facts and a range of algorithms for computation with all numbers. Together, these understandings and competencies will provide access to the other content of measurement, geometry, statistics and probability, as well as to complex problem solving, all of which use and generalise the basic concepts and processes developed with number. The relationship between the proficiency strands and the content strands can be summarised as shown in the following table.

MATHEMATICS			
	NUMBER AND ALGEBRA	MEASUREMENT AND GEOMETRY	STATISTICS AND PROBABILITY
Understanding	numeration additive and multiplicative thinking algorithmic thinking algebraic thinking	units of measurement shape geometric reasoning location and transformation	data representation and interpretation chance
Fluency			
Problem solving			
Reasoning			

This summary of the components of the *Australian Curriculum: Mathematics* framework has been designed to highlight several important aspects that need to be considered. The relevant space in the diagram given to each proficiency strand is designed to show the following:

- The development of *conceptual understanding* needs to be established prior to further development of a topic or content area, as it is critical to building up fluency and problem solving.
- *Fluency* means a process can be accessed and used efficiently and accurately almost without thinking of the steps involved, so that thought can be given to the way it is used rather than to the procedures to be followed in its use.

- *Problem solving* does not arise simply as a consequence of understanding and fluency. Time and resources need to be allocated as much to establishing problem solving as to building competence with computational and other processes.
- *Reasoning* will be developed along with understanding, fluency and problem solving, but there also needs to be time devoted to ensuring that students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising.

Within Number and Algebra, *numeration* (understanding number) is critical; addition and subtraction are grouped as *additive thinking*; multiplication and division are brought together as *multiplicative thinking*; and emphasis on the thinking underpinning numeration and computational processes is viewed as *algorithmic thinking*, a way of thinking that underpins all higher mathematics, rather than simply as procedures for obtaining answers that can be learned in isolation. Similarly, the components of Measurement and Geometry, Statistics and Probability, drawn from the content descriptions, need to be seen as linked together with Number and Algebra to form a connected and coherent view of the mathematics to be learned, rather than as distinct topics, each with different procedures and ways of operating.

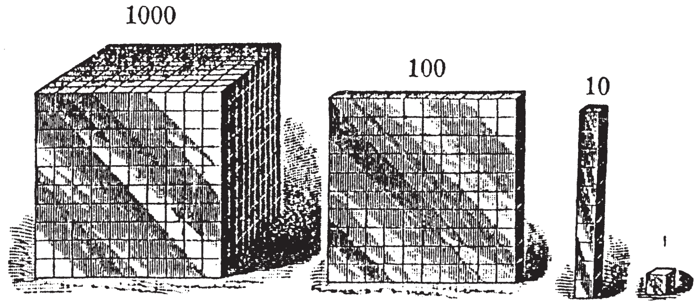
As Hiebert and Grouws (2007) noted in their synthesis of international research calling for a more detailed, richer and coherent knowledge base to inform practice:

> Two features of classroom teaching facilitate students' conceptual development (and mathematical proficiency), explicit attention to connections among ideas, facts and processes, and engagement of students struggling with mathematics. (Hiebert & Grouws, 2007, p. 391)

The *Australian Curriculum: Mathematics* has been designed to build students' conceptual understanding, and thus fluency, by attending explicitly to the connections among all aspects of the mathematical content. As teachers, we need to ensure that students engage with the mathematics they are learning, and that this mathematics is not simply focused on routines that are readily known, but provides a move towards deeper and more powerful knowledge that challenges all students at all levels. Struggling to come to terms with this mathematics will then become a natural and enjoyable part of mathematics learning.

An essential part of teaching mathematics is to provide meaningful experiences at appropriate points out of which appreciation and understanding of concepts and ways of thinking can be built. Such activities can be meaningful in the teaching of mathematics only when two interrelated aspects are present. The meaning inherent in the situation, materials, models or diagrams, patterns, language or symbols through which the new notions are being expressed must match the mathematical meaning that is intended to be built up. These meanings must also match the children's level of development and ability to take in information, to generalise and construct a reasonable view of the underlying mathematics. In other words, experiences need to be meaningful for both the mathematics and the child.

Materials in use today to represent numbers, such as ice cream sticks bundled to show tens and ones, not only provide a structure that the children can see, but also match identically the behaviour of the number system itself. Such materials have a long history, from the use of bead frames to show tens and hundreds popularised by Maria Montessori in the early 1900s to the use of blocks to show the ones, tens, hundreds and thousands promoted in arithmetic books of the 19th century.



Fish's New Arithmetical Series, 1883

Nonetheless, as Cobb, Yackel and Wood observed, 'manipulative materials can play a central role if we wish students to learn with understanding, but the way the materials are interpreted and acted upon must necessarily be negotiated by the teacher and students' (1992, p. 7). The materials are not 'transparent' representations of a readily apprehensible mathematics, but instead are vehicles for the potential meanings that children might construct.

1.3

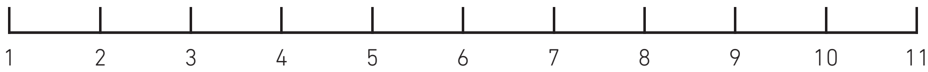
Explain the importance of using materials to develop mathematical concepts and processes

Using materials to develop mathematics

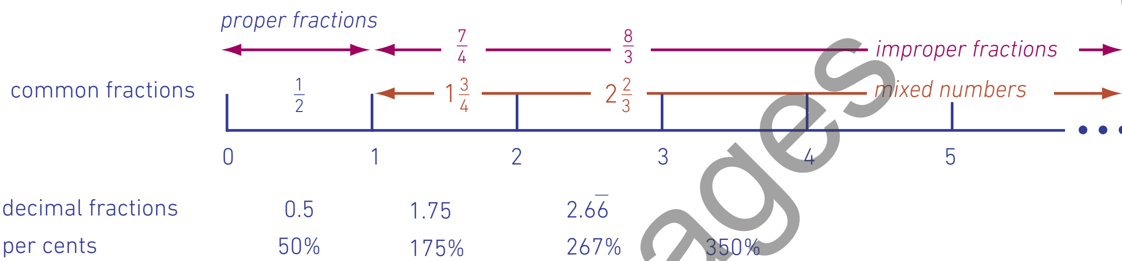
The need for materials and models is fundamental in teaching mathematics because so many of the ideas that have to be learned are not intrinsically obvious. They were generalised and developed from diverse and obscure situations over a long period of time, usually by mature thinkers who had particular social or intellectual needs. If young children are to be assisted to develop the same forms of thinking, then situations in which these ideas can be discerned and discussed are essential. It is for this reason that the teaching of mathematics has had a long tradition of using structured materials, materials through which the underlying mathematical ideas might come to be perceived and appreciated. For instance, ten frames and bundling sticks are used to establish early numbers, Base 10 materials show the place value patterns for larger numbers, and region models give meaning to fraction ideas. Even comparatively simple notions such as the initial number concepts are abstractions rather than something that can be seen in the immediate environment. The number nine does not exist in the real world, but is a representation that children need to construct for themselves by linking objects with language and symbols within meaningful experiences with materials that sometimes show and sometimes do not show the concept of *nineness.* Similarly, the basic spatial forms can only take on meaning through seeing and making representations of the general notion of what will come to be called a rectangle or a triangle and learning to distinguish one from the other.

At the same time, it must be borne in mind that materials by themselves do not literally carry mathematical meaning. While they might assist in the initial building of understanding, it is reflection on the actions of the materials and the situations that they represent that allows the generalisation to a mathematical way of thinking, rather than a rote learning of the results of these actions. If children focus solely on the outcomes, it is possible that they will simply learn at a surface level how to manipulate the materials rather than the deeper, fundamental mathematical ideas. This risk becomes even larger if the materials come to be seen as ends in themselves, rather than as links to the mathematical concepts and processes they are intended to represent.

For instance, many children are introduced to a number line early in their learning of number concepts, using 'hops' or 'jumps' forwards or backwards on it in order to *count on* or *back, add* or *subtract*. Often such a number line does not include zero, but starts at 1:

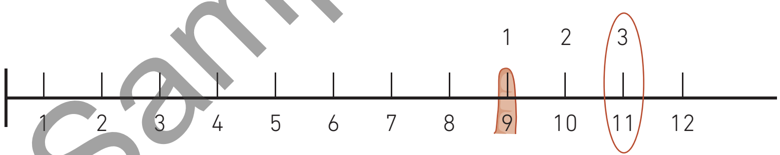


However, use of the number line as a means to find answers leads to children seeing only the points on a line, rather than the fact that the numbers actually represent the *distances from zero*. When fraction ideas are introduced, a number line is essential to show how these new numbers occur among the whole numbers based on the distances between the whole numbers.



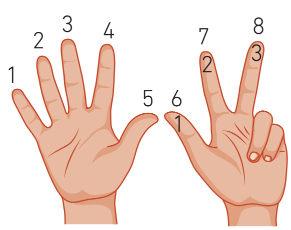
A number line shows that fractions are parts of 1 one, 2 ones, 3 ones, … and names them as proper and improper fractions, mixed numbers, decimal fractions and per cents

The early use of points on a number line also leads children to focus on the result, rather than on the thinking that supports the result—for instance, counting 1, 2, 3 when counting on from 9 instead of 10, 11, 12. This often results in an incorrect result of 9 when the counting begins at 9 rather than with the next number, 10:



Child: *The answer is 11.*

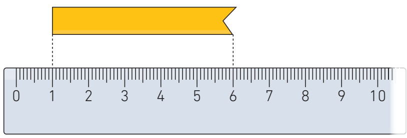
Similarly, when adding 6 and 3, they may count 1, 2, 3, 4, 5, 6, then a further 1, 2, 3 to get to an answer which is then determined by counting all the way from 1 to 9. Since these approaches are hard to keep track of mentally when the number line is no longer available, fingers are often used to obtain the answers, and then become the main means of responding to addition and subtraction basic fact questions or when they are needed in other processes.



Child: *The answer is 1, 2, 3, 4, 5, 6, 7,* ***8***.
With the 6th finger counted twice, the child has answered 8; whereas counting on from 6 would give 7, 8, **9**.

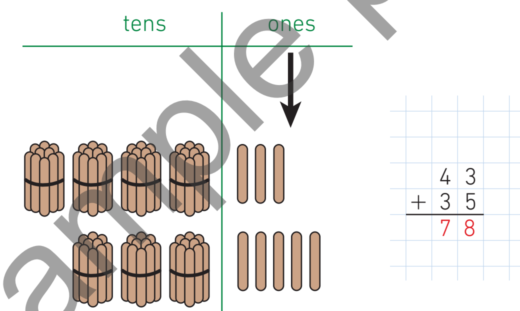
Moreover, when counting, children will often count using the last number rather than beginning with the next number, so that in counting on and addition their answers are 1 too small, while for counting back and subtraction their answers are 1 too large. They may count to 6, then begin at 6 to count a further 1, 2, 3 to get to an answer of 8, or count back 3 from 10 to get an answer of 8 by saying 10, 9, **8**.

Use of the number line with a focus on the counting numbers 1, 2, 3, . . . also leads to children's difficulties with initial measurement techniques. If zero is not integrated into their conception of numbers from the beginning, they assume that the initial mark to be used on a ruler is the 1 and then find that all of their measurements are one unit too short.

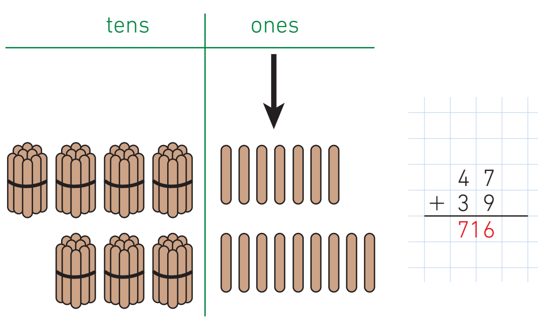


Child: *The length of the ribbon is 6 cm.*
Actual length should be 5 cm when measured from 0, not from 1.

Another common example of when children experience difficulties because of the way they see manipulating material as an end in itself is when bundling sticks or Base 10 materials are used to introduce the addition algorithm. When the first examples given involve only simple combinations such as 43 + 35, a correct answer can be given by simply putting the tens and then the ones together.

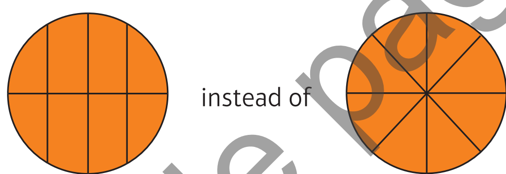


Yet, later on, it will be necessary to join the ones before the tens in order to cope with additional tens that might result. With the materials, this can be accounted for by rearranging the sticks that result. In the written form, it needs to occur as the algorithm proceeds or else there is a danger that an example like 47 + 39 is answered with 716, rather than 86.



Thus, materials by themselves will not be sufficient and their use needs to actively engage children's thinking with guidance from teachers and other users as they build towards long term outcomes of which the learner may initially have little awareness.

Materials are not something to be used only at the beginning of mathematics teaching and learning. Concepts and processes need to be introduced through realistic problem settings at all levels, and materials offer a very feasible way of portraying both the problem and its possible solutions, leading to the generalisations which constitute higher mathematics. At the same time, there is a need to guard against the use of *ad hoc* exemplars—that is, materials that seem to assist the task at hand but that in the longer run prove to be inadequate for further developments. Rather, materials that can be used for many different purposes can assist in building a connected view of mathematics. There would seem to be advantages in using the same representation in different situations, rather than using different representations in the same situation (Clements & McMillen, 1996). The teaching of fraction ideas is perhaps the most compelling topic where this has been overlooked. Models involving regions of a circle show the parts within a whole but do not allow children readily to show for themselves the basic part/whole conception. Indeed, many children choose the length across the diameter of the circle on which to measure off equal parts, rather than determine same sized portions of the area of the circle.



A circle partitioned to show 8 (equal) parts

Another difficulty is that these circular models cannot readily be extended to decimal fractions or per cents, yet these are all the same fundamental fraction idea.

Similarly, when materials based on regional models are used to portray addition, a child may have difficulty interpreting the diagrams used.

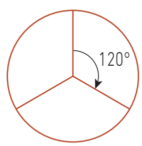
3 parts out of 4 parts + 3 parts out of 4 parts is 6 parts out of 8 parts?

It is even more difficult to envisage how these models would be able to show subtraction, let alone multiplication or division.

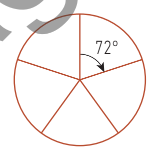
While some materials would seem to suit certain purposes well, other situations demand different models. Yet, if there are too many different models for a given concept, children will need to be operating at a formal level of cognition to bring the diverse representations together into the one understanding. It is more likely that they will see each material or model as a distinct entity and try to learn many fragmented rules. Thus, children who have been introduced to multiplication as *groups* or *steps on a number line* and in terms of *area* tend to view each separately with its own ways of proceeding. Consequently, they end up with no firm basis of multiplication at all. In contrast, if the use of materials at first focuses on one strong model or material, such as an *array* for multiplication, not only will this give security to the initial conception, but it will provide a basis for future development to all of the possible representations by reference back to a secure, fundamental idea.

It is rather analogous to the manner in which a plant grows; at first a strong central stem is produced, and only when this is secure do other limbs branch off. Where many stems emerge at the same time, competition ensures that there is no single branch strong enough to last on its own and frequently the whole plant simply withers and dies. Thus, wherever possible the material or model that will assist the end point of development should be the one that introduces the initial idea and be the backbone of the development from beginning to end. If the same materials can also be used for different but related aspects of mathematics, so much the better. This will be of invaluable assistance in allowing a cohesive, connected view of mathematics and its applications to be developed.

A further difficulty with many of the materials and models proposed for classroom use is that they actually demand knowledgeable users. It is the experience and understanding of the proponent that carries the mathematics, rather than the materials themselves. The use of circular regions for common fractions, mentioned earlier, is an instance of this. In reality, it is the angular measure about the centre that determines the number of parts, yet children often meet and use these models some time before they come to terms with angles as such.



Each angle is 120°



Each angle is 72°

Another instance is when the values of the Base 10 materials used to show hundreds, tens and ones are suddenly altered to show decimal fractions. It is important that the materials and models themselves are able to portray the underlying ideas so that constant interjection and explanation by the teacher who is using them is not necessary. Otherwise, the materials or models may simply be a distracter and the children resort to learning how to manipulate them at a surface level instead of coming to terms with the ideas that are supposed to be portrayed.

Materials and models are fundamental to learning mathematics in all forms and at almost all levels, although their nature and the ways in which they are referred to may change. At some point they may need to be very concrete, able to be picked up, pulled apart or manipulated. The use of such materials can build in from the outset that mathematics is fundamentally an experimental science; conjectures need to be made and evaluated in place of simply acquiring a mass of seemingly immutable facts and rules. Mathematics at all levels is a developing body of knowledge. Activities inherent in the use of materials and models can introduce that way of thinking from the very beginnings of its development in each child. Such materials will include Base 10 blocks to represent numbers and calculations; clocks, money, rulers, balances and other instruments for measurement; solid objects, drawing instruments and online maps for geometry; and physical graphs and the use of dice or spinners for statistics and probability.

At a later time, a pictorial reference may suffice. Actual objects may give way to pictures or diagrams of them to represent numbers and computations; computer programs may simulate measurement, geometry, and statistics and probability; and diagrams formed *in one's head* may replace pencil and paper drawings for problem solving across all areas of mathematics. Indeed, a reference to the experiences that were produced with materials may be all that is needed to allow the development of further ideas.

Language and mathematics

While the role of materials, models and the patterns they develop is fundamental, materials by themselves do not literally carry meaning. Their most important use is to provide experiences that can be discussed and reflected on to allow the effects of the embodied actions to emerge as mathematical ways of thinking. It is language that communicates ideas, not only describing concepts but also helping them to take shape in each learner's mind (Usiskin, 1996). Discussion among the participants is needed to bring out 'the explicit construction of links between understood actions on the objects and related symbol processes' (Ma, 1999, p. 20). In building problem solving abilities, students need to report, display, explain and argue for their own solutions to see that 'getting the right answer is only the beginning rather the end . . . the ability to communicate thinking convincingly is equally important' (Schoenfeld, 2001, p. 54).

Language is the key to all aspects of mathematics learning, from the formation of concepts and processes, through problem solving, to the development of numerate students who will thrive in the technology dominated world in which they will live. However, as teachers and students attempt to come to terms with their mathematical understandings, the meanings given and taken may differ not only from one another but also from the intended mathematics (Pirie, 1998). Care is needed that the language used matches the materials and experiences provided, the language levels of the learners, and the mathematical concepts and processes being addressed.

1.4

Recognise the role of language in teaching and learning mathematics

<table>
<tr><td colspan="3">Language use with addition: and or add?</td></tr>
<tr>
<td>At first, and is used with its reference to 'joining' (the initial meaning of addition).

4   4
+3   and   3
7   is   7</td>
<td>When addition of 2 digit numbers is introduced, and is best used.

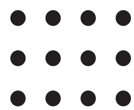
47   47
+38   and   38
85   is   85</td>
<td>When addition of 3 digit numbers is introduced, the use of and should give way to add.

476   4 hundred and 76
+389   add 3 hundred and 89
865   is   8 hundred and 65</td>
</tr>
<tr>
<td colspan="2">Use and to signify addition as joining.</td>
<td>At this point, using add is preferable to emphasise the operation of addition and to avoid confusion with the way the numbers are read using and.</td>
</tr>
</table>

Yet, mathematics classrooms were once envisaged as silent places and communication from one child to another about the mathematics they were trying to come to terms with was discouraged, seen as a form of cheating. Now, talking about the ideas that arise is encouraged, and cooperation and communication are desired behaviours rather than forbidden practices. When mathematical ideas are communicated, particular care is taken in formulating language to keep track of what is happening with materials or representations, which will eventually allow formal symbolic recording, mental operations or approximations to be conducted confidently.

Throughout *Teaching Primary Mathematics*, this aspect of the development of mathematics is fundamental, with particular attention paid to the use of materials and patterns to engender an appropriate way of governing the more formal symbolic expressions of mathematical processes through the use of a language that at first describes and then directs the thinking that is involved. Care has been taken in establishing uses of materials to lay the foundation for particular processes and in using a form of language at a level appropriate to both the learner and the mathematics. The same remarks about meaning that were applied to the use of materials are also pertinent to the language that can be encouraged to emerge from the concrete experiences. As mathematical ideas are constructed, so too must be the language that enables them to become the individual learner's own. In turn, this leads to the thinking that ensures each child possesses control over the processes. Consistency in this use of language is extremely important. This does not necessarily mean that individual words need to be used in preference to others; rather, that their meaning is available to the particular group of children and is consistent with the use of materials, the patterns that emerge, and from one situation to the next.

Does 3 times 4 mean 4 x 4 x 4 or 3 x 3 x 3 x 3, or both? 3 fours and 4 threes are much clearer and can be shown to have the same meaning through the use of an *array*

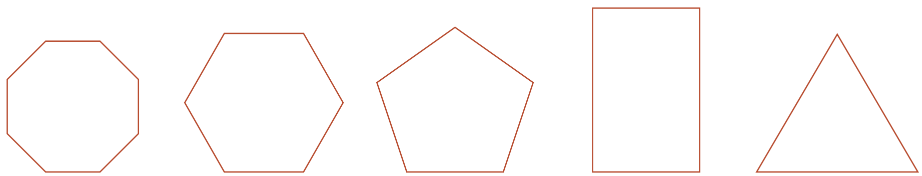


Mathematics has often been called a symbolic language, reflecting the importance given to the concise symbols used to stand for concepts that in time allow these ideas to be manipulated easily. Yet, mathematics is both oral and written, and language is needed to communicate the meaning of symbolic statements. For example, a student is often able to say that '3^4 is 3 to the power 4', but when pressed they have no meaning at all for these words and may be unable to talk about 3 being a *factor 4 times* or that 3^4 means $3 \times 3 \times 3 \times 3$, or 81. Similarly, while the symbol π provides a concise way of stating the ratio of circumference to diameter of a circle, unless the symbol can be verbalised it is unlikely that any meaning will be attached to it or to the various formulas, such as πr^2 for the area of a circle or $2\pi r$ for circumference, in which it is used. Indeed, it is through using lengthy informal statements that sufficient understanding gives rise to a need for more succinct specialised words as well as for symbols to express mathematical ideas.

This specialised vocabulary of mathematics can be as concise and dense as the mathematical symbols with which we most readily associate the writing of ideas in mathematics, and should be left until the underlying meaning is quite clear. There needs to be a careful building up of experiences and descriptions on which new mathematical words are based, and care needs to be taken in introducing these words by reference to the underlying meaning. Since many of the words that are used have entered mathematics from Greek, Latin or Arabic sources, investigating their original meanings can also be helpful to children. For instance, many children have difficulties when asked to determine the perimeter of a shape, but can readily measure the distance all the way around the outside of the shape. There is often a distinction between knowing the meaning of a concept and being able to recall and use the corresponding mathematical vocabulary. In fact, *perimeter* is made up of Greek words that literally mean to *measure around the outside*. It is helpful to children to realise that we use the Latin expression with the same meaning, *circumference*, when we measure the distance *around the outside of a circle*.

Other words in measurement and geometry also have origins that can help children move from an informal description to adopt the more formal mathematical names. Some shapes are named in terms of the angles they contain, using Greek number words for prefixes together with part of the word *gonia*, which originally meant 'corner' and expanded to indicate 'angle', as in *pentagon* (5 angles), *hexagon* (6 angles), and so on. Others are named using Latin prefixes, as in *triangle* (3 angles) and *rectangle* (from words that mean 'right angle', hence 4 right angles), or are named from Latin words that refer to sides, as in *quadrilateral* or *equilateral*. Interestingly, a triangle in ancient Greek was *trigon*, which gave its name to the branch of mathematics called *trigonometry*, while a 4 sided shape was called a *tetragon* before it was called a *quadrilateral*, thus giving rise to the *quadratic equation*, which was concerned with 'squared' amounts.

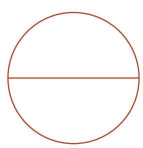
Gon comes from the Greek word *gonia*, which means 'angle'.



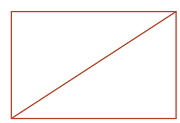
octagon 8 angles | hexagon 6 angles | pentagon 5 angles | tetragon (quadrilateral) 4 angles | trigon (triangle) 3 angles

Having some knowledge of the counting words in Latin and Greek can help with many of the seemingly more complex words that are used in mathematics to name shapes, large numbers, and even properties such as bisect or trisect. Knowing that *dia* means 'across' can help with the meaning of *diameter* (a measure across a circle) and *diagonal* (a line from one vertex to another), while understanding that *isos* means 'same' can help fix the meaning of an *isosceles* triangle.

dia means 'across'.



Diameter: a measure across a circle



Diagonal: a line across from one vertex to another

Treating the words of mathematics as parts of a specialised language can lead to building up the meanings for children in the same way that word meanings are built up in a first language, rather than assuming that they can be readily tagged to quite complex mathematical ideas. A very good source for this information is a book published by the Mathematical Association of America on the etymology of mathematical terms (Schwartzman, 1994). Linking to the sites *History of Mathematical Symbolism* (http://jeff560.tripod.com/mathsym.html), *Terminology* (http://jeff560.tripod.com/mathword.html) and the more general *etymology* site (www.etymonline.com) and searching 'mathematics' will quickly provide the meaning and origin of words and symbols used in mathematics.

Another way in which the notion of language enters mathematics is through the use of words that have subtly different meanings from those exhibited in their everyday use. For instance, some words have their meaning expanded; in everyday use, the word 'more' means to increase, but in many mathematics situations it means to find the difference, as in: *I have 4 oranges, you have 5 oranges. How many more oranges do you have than I have?* Similarly, in everyday use the word 'rectangle' refers to a shape in which one side is longer than the other, while a 'square' has all sides of equal length and a 'curve' is a line in the form of a bend, with no straight part. In mathematics, the concept of rectangle includes squares, since each contains four right angles, and the notion of a curve includes straight lines. Other words have their meaning contracted: the word 'product'